HOLONOMIC - DEPENDS ONLY ON $\{\vec{r}_i\}, t$

WE CAN WRITE

$$f(\vec{r}_1, \vec{r}_2 \dots \vec{r}_n, t) = 0$$

$$f(\{\vec{r}_i\}, t) = 0$$

PARTICLE ON A PLANE $x=5$

PARTICLE ON AN ELLIPSE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - c^2 = 0$$

NON HOLONOMIC

DEPEND ON v OR INEQUALITY

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} < c^2$$

RHEONOMOUS - TIME DEPENDENT

SCLERONOMOUS - TIME INDEPENDENT

$\{\vec{r}_i\}$ NO LONGER INDEPENDENT

$\exists$ FORCES OF CONSTRAINT

THERE EXISTS

(WE KNOW THE RESULTS OF THE CONSTRAINT, BUT NOT THE FORCES CAUSING THEM)

HOLONOMIC $\Longrightarrow$ USE GENERALIZED COORDS

N PARTICLES $\Longrightarrow$ 3N COORDS

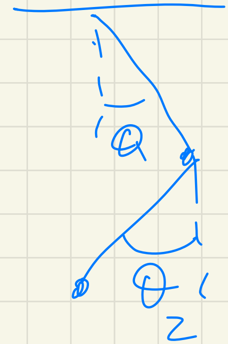
K CONSTRAINTS

$\Longrightarrow$ $3N-K$ DEGREES OF FREEDOM

$$\vec{r}_i = \vec{r}_i(q_1, q_2 \cdots q_{3N-K}, t)$$

PARTICLE ON SPHERE: LATITUDE
LONGITUDE

DOUBLE PENDULUM



θ_1, θ_2

NONHOLONOMIC
- CANNOT ELIMINATE VARIABLES
- MORE COMPLICATED

DISC ROLLING
1D: HOLONOMIC
2D: NON HOLONOMIC

VIRTUAL INFINITESIMAL DISPLACEMENT

$\delta \vec{r}_i$

SYSTEM IN EQUILIBRIUM $\vec{F}_i = 0$

$\vec{F}_i \cdot \delta \vec{r}_i = 0$ ZERO VIRTUAL WORK

$$\sum_i \vec{F}_i^a \cdot \delta \vec{r}_i + \sum_i \vec{f}_i \cdot \delta \vec{r}_i = 0$$

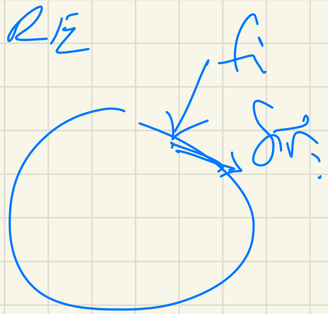
APPLIED FORCE FORCE OF CONSTRAINT

CHOOSE SYSTEMS $\ni$ CONSTRAINT

SUCH THAT

FORCES DO ZERO WORK.

PARTICLE ON SPHERE



$$\sum_i \vec{F}_i^a \cdot \delta \vec{r}_i = \sum_i \vec{f}_i \cdot \delta \vec{r}_i$$

$= 0$

$$\sum_i \vec{F}_i^a \cdot \delta \vec{r}_i = 0$$

$\{\delta \vec{r}_i\}$ ARE NOT INDEPENDENT

NON EQUILIBRIUM

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$$

$\vec{F} = \dot{\vec{p}}$ $\qquad$ $\vec{F}_i = \vec{F}_i^a + \vec{f}_i$

$$\sum_i (\vec{F}_i^a - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$$

BECAUSE $\sum_i \vec{f}_i \cdot \delta \vec{r}_i = 0$

NOW TRANSFORM TO GENERALIZED COORDS

$$\vec{r}_i = \vec{r}_i(q_1 \cdots q_n, t) \qquad \dot{q}_k = \frac{dq_k}{dt}$$

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \quad 1$$

$$\Rightarrow \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$

$$\delta\vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j$$

1st TERM $\sum_i \vec{F}_i \cdot \delta\vec{r}_i$ $\qquad F_i = F_i^a$

$$= \sum_j F_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j$$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$ GENERALIZED FORCE

$$\sum_i \dot{\vec{P}}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right]$$

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \frac{\partial}{\partial q_j} \frac{d\vec{r}_i}{dt}$$

1

$$= \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j q_k} \dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}$$

$$= \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right. \left. - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j}\right]$$

$$\Rightarrow \sum_i \left(\vec{F}_i^a - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0$$

$$0 = \sum_j \left\{ \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right] - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right) - Q_j \right\} \delta q_j$$

IF CONSTRAINTS HOLONOMIC,

CHOOSE $\{q_j\}$ INDEPENDENT

$$\boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j}$$

IF FORCES ARE CONSERVATIVE

AND $\vec{F}_i = -\vec{\nabla}_i V$ (IE: NO VECTOR POTENTIAL)

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$= -\frac{\partial V}{\partial q_j}$$

$$\frac{dV}{dx}\frac{\partial x}{\partial q_j} + \frac{dV}{dy}\frac{\partial y}{\partial q_j} + \frac{dV}{dz}\frac{\partial z}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

IF V ONLY DEPENDS ON q_j

AND NOT $\dot{q}_j$ $\quad \frac{\partial V}{\partial \dot{q}_j} = 0$

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

$$\boxed{\mathcal{L} = T - V}$$

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0}$$