$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$y' = \frac{dy}{dx} \qquad J = \int_{x_1}^{x_2} f(y, y'; x)\, dx$$

J stationary $\Rightarrow \frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$

a) minimum distance between 2 points

$$ds = \sqrt{dx^2 + dy^2} \qquad J = \int_1^2 ds$$

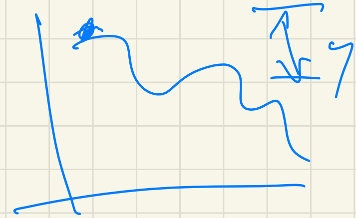
$$J = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad f = \sqrt{1 + y'^2}$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$\Rightarrow y'$ INDEPENDENT OF x

$y' =$ constant $\qquad y = ax + b$

Minimum time curve for a sliding object



y measured down

$$\frac{1}{2}mv^2 = mgy$$

$$v = \sqrt{2gy}$$

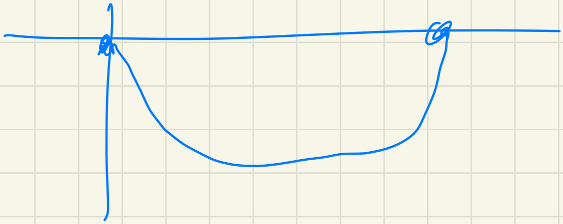
$$t = \int dt = \int \frac{ds}{v} = \int_1^2 \overbrace{\frac{\sqrt{1+y'^2}}{\sqrt{2gy}}}^{f}\,dx$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$$

skipping ~~lots~~ lots of steps

$$x = a(\phi - \sin\phi)$$

$$y = a(1 - \cos\phi)$$



$$\int_{t_1}^{t_2} dt \sum_{i=1}^{n} \underbrace{\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i}\right]}\delta q_i = 0$$

GETTING $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$

REQUIRES $\{\delta q_i\}$ ARE INDEPENDENT

CONSTRAINTS $\Rightarrow$ NOT INDEP.

$$f_j(\{q_i\}) = 0 \qquad j = 1, \dots, k$$

f CAN ALSO BE TIME DEP.

$$0 = \delta f_j = \frac{\partial f_j}{\partial q_1}\delta q_1 + \dots \frac{\partial f_j}{\partial q_n}\delta q_n$$

$$= \sum_{i=1}^{n} \frac{\partial f_j}{\partial q_i}\delta q_i$$

$$0 = \lambda_j \sum_{i=1}^{n} \frac{\partial f_j}{\partial q_i}\delta q_i$$

$$\sum_{j=1}^{k} \lambda_j \sum_{i=1}^{n} \frac{\partial f_j}{\partial q_i} \delta q_i = 0$$

Integrate over t

$$\int_{t_1}^{t_2} dt \left(\sum_{j=1}^{k} \sum_{i=1}^{n} \lambda_j \frac{\partial f_j}{\partial q_i} \delta q_i \right) = 0$$

Add to Lagrange eq

$$\int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i} \right) \delta q_i = 0$$

$\{\delta q_i\}$ n of these

$n-k$ independent

use

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i} = 0$$

use k of these to get $\{\lambda_j\}$

makes $\{\delta q_i\}$ effectively independent

n equations
k constraint eq
$n+k$ variables $\{q_i\}_{(n)}$ $\{\lambda_j\}_{(k)}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i}$$

REWRITE WITH T

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i + Q_i^{\text{CONST}}$$

CONSTRAINT

FORCES

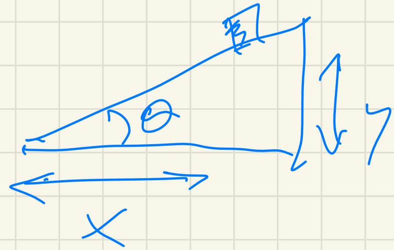
$$Q_i = -\frac{\partial V}{\partial q_i} \qquad \frac{\partial V}{\partial \dot{q}_i} = 0$$

(FOR NOW)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i^{\text{const}}$$

$$Q_i^{\text{const}} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i}$$

MASS SLIDING DOWN AN INCLINED PLANE



$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$f(x,y) = y - x\tan\theta = 0$$

$$\mathcal{L} \to \mathcal{L} + \lambda f = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(y - x\tan\theta)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} = \frac{\partial \mathcal{L}}{\partial x} = 0 - \lambda\tan\theta$$

$$\ddot{x} = \frac{-\lambda\tan\theta}{m}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\ddot{y} = \frac{\partial \mathcal{L}}{\partial y} = -mg + \lambda$$

use $f \Rightarrow y = x\tan\theta \quad \ddot{y} = \ddot{x}\tan\theta$

$$\frac{m\ddot{y}}{\tan\theta} = -\lambda\tan\theta \qquad -\lambda\tan^2\theta = -mg + \lambda$$

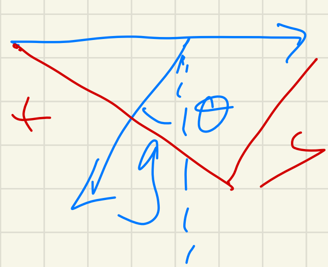
$$\lambda(1 + \tan^2\theta) = mg$$

$$\Rightarrow \lambda = mg\cos^2\theta$$

$$\ddot{y} = -g + g\cos^2\theta = -g\sin^2\theta$$

$$\ddot{x} = -g\sin\theta\cos\theta$$

$$\left\{\begin{array}{l} a_{x'} = g\sin\theta \\ x = x'\cos\theta \\ y = x'\sin\theta \end{array}\right.$$



CHECK WITH TILTED SOLUTION

CONSTRAINT FORCES

$$F_x = \lambda \frac{\partial f}{\partial x}$$

$$y - x\tan\theta = 0$$

$$\lambda = mg\cos^2\theta$$

$$= mg\cos^2\theta(-\tan\theta)$$

$$= -mg\cos\theta\sin\theta \quad (= m\ddot{x})$$

$$F_y = \lambda \frac{\partial f}{\partial y} = mg\cos^2\theta \quad (\neq m\ddot{y})$$

SEMIHOLONOMIC CONSTRAINT

$$f_\alpha(\{q_i\}, \{\dot{q}_i\}, t) = 0 \qquad \alpha = 1, \dots m$$

f_α linear in $\{\dot{q}_i\}$

$$f_\alpha = \sum_{k=1}^{n} a_{\alpha k}(\{q_i\}, t)\,\dot{q}_k + a_0(\{q_i\}, t) = 0$$

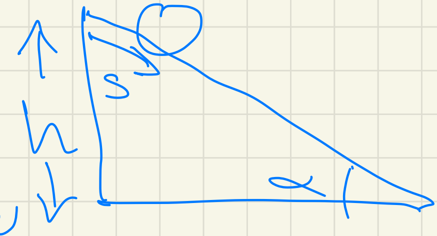
$$\Rightarrow \delta \int_{t_1}^{t_2} \left(\mathcal{L} + \sum_{\alpha=1}^{m} \mu_\alpha f_\alpha \right) dt = 0$$

μ_α = Lagrange multipliers

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k = \sum_{\alpha=1}^{m} \mu_\alpha \frac{\partial f_\alpha}{\partial \dot{q}_k}}$$

VELOCITY DEPENDENT CONSTRAINTS

HOOP ROLLING DOWN INCLINE



θ = ANGLE OF ROTATION OF HOOP

s = DISTANCE TRAVELLED $= \sqrt{x^2+y^2}$

$s = R\theta$ HOLONOMIC

$\Rightarrow \dot{s} - R\dot{\theta} = 0$ NON HOLONOMIC

$$\mathcal{L} = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}I\dot{\theta}^2 - (h - s\sin\alpha)mg$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{s}} - \frac{\partial \mathcal{L}}{\partial s} = -\lambda\frac{\partial f}{\partial \dot{s}} \qquad I = mR^2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = -\lambda\frac{\partial f}{\partial \dot{\theta}}$$

$$\left[\begin{array}{l} m\ddot{s} - mg\sin\alpha + \lambda = 0 \\ mR^2\ddot{\theta} - \lambda R = 0 \\ R\dot{\theta} = \dot{s} \end{array}\right.$$

3 eq

3 unknowns

$$\left\{\begin{array}{l} m\ddot{s} - mg\sin\alpha + \lambda = 0 \\ mR^2\ddot{\theta} - \lambda R = 0 \\ R\dot{\theta} = \dot{s} \implies R\ddot{\theta} = \ddot{s} \end{array}\right.$$

$$\lambda = mR\ddot{\theta} = m\ddot{s}$$

$$\ddot{s} = \frac{g\sin\alpha}{2}$$

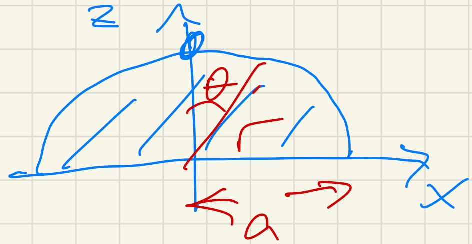
$$\lambda = \frac{mg\sin\alpha}{2}$$

FRICTIONAL FORCE ON RIM OF HOOP

$$\ddot{\theta} = \frac{g\sin\alpha}{2R}$$

HOLONOMIC ?

?



$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz$$

$$f = a - \sqrt{x^2+z^2} = 0 \qquad f = a - r = 0$$

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\cos\theta$$

$$x = r\sin\theta \qquad z = r\cos\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = -\lambda\frac{\partial(a-r)}{\partial r}$$

$$\cancel{m\ddot{r}} - mr\dot{\theta}^2 + mg\cos\theta = \lambda$$

$$\cancel{2m\dot{r}\dot{\theta}} + mr^2\ddot{\theta} - mgr\sin\theta = 0$$

APPLY CONSTRAINTS

$$\ddot{\theta} = \frac{g\sin\theta}{a} \qquad ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$\ddot{\theta} = \frac{g \sin\theta}{a}$ $\qquad ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0$

SOLUTION FROM BOOK $\qquad \dot{\theta}^2 = -\frac{2g}{a}\cos\theta + \frac{2g}{a}$

$$\lambda = mg(3\cos\theta - 2)$$

CHECK SOL'N : $\quad 2\cancel{\dot{\theta}}\ddot{\theta} = \frac{2g}{a}\sin\theta\,\cancel{\dot{\theta}}$ ✓

λ POSITIVE $\Rightarrow$ CONSTRAINT FORCE

$\Rightarrow \quad 3\cos\theta \geq 2$

MASS FLIES OFF THE DOME AT $\cos\theta = 2/3$