NOETHER THM – Any Transformation that leaves the laws of Nature (eq of motion) unchanged

$\Longleftrightarrow$

Conserved quantity

t independence $\Longleftrightarrow$ E conservation

x independence $\Longleftrightarrow$ p_x "

θ " $\Longleftrightarrow$ L "

Assume $\bar{V} = \bar{V}(\{q_i\})$ ONLY

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \bar{T}}{\partial \dot{x}_i} - \frac{\partial \bar{V}}{\partial \dot{x}_i} = m\dot{x}_i = P_{x_i}$$

GENERALIZED MOMENTUM

$$\boxed{P_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}}$$

CANONICAL

OR

CONJUGATE

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \underbrace{\frac{d}{dt} p_{x_i}}_{\dot{p}_{x_i}} = \frac{\partial \mathcal{L}}{\partial q_i} = \underbrace{-\frac{\partial V}{\partial q_i}}_{F_i}$$

$$\dot{p}_{x_i} = F_i$$

EXAMPLE: EM FIELD

$$\mathcal{L} = \frac{1}{2} m r^2 + q\phi + q\vec{A}\cdot\vec{r}$$

$$\vec{A} = \vec{A}(\vec{r})$$

$$\phi = \phi(\vec{r})$$

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + qA_x$$

IF $\mathcal{L}$ IS INDEPENDENT OF q_k

q_k IS "CYCLIC" OR "IGNORABLE"

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \dot{p}_k = \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

$\Rightarrow p_k$ CONSERVED (CONSTANT)

EXAMPLE $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y)$

(NOT z)

$$\frac{\partial \mathcal{L}}{\partial z} = 0$$

$\Rightarrow p_z$ CONSERVED

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad p_y = m\dot{y}$$

$$p_z = m\dot{z}$$

2D MOTION IN A CENTRAL POTENTIAL

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

not ϕ

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow P_\phi \quad \text{CONSERVED}$$

$$P_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi} = L_z \quad \text{CONSTANT}$$

$$P_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \quad \text{NOT CONSERVED}$$

$$\dot{\phi} = \frac{P_\phi}{mr^2}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0$$

$$m\ddot{r} = \underbrace{mr\dot{\phi}^2}_{\text{CENTRIPETAL}} - \underbrace{\frac{\partial V}{\partial r}}_{F} = \underline{\frac{P_\phi^2}{mr^3}} - \frac{\partial V}{\partial r}$$

WE DERIVED MOMENTUM CONS EARLIER FROM 3^{RD} LAW.

THIS P CONS IS A STRONGER RESULT BECAUSE IT ONLY USES POSITION INVARIANCE

EXAMPLE $\mathcal{L} = \frac{1}{2} m \dot{r}^2 + q\phi + q\vec{A} \cdot \vec{r}$

$P_x = m\dot{x} + qA_x$

If $\phi \neq \phi(x)$ AND $\vec{A} \neq \vec{A}(x)$

THEN P_x IS CONSERVED

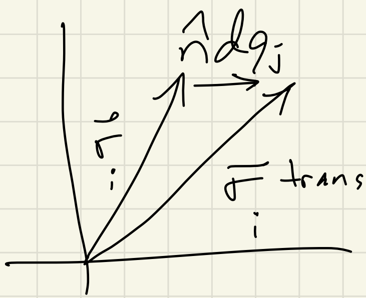
CONSIDER A TRANSLATION OF THE ENTIRE SYSTEM BY dq_j

T INDEPENDENT OF q_j $\quad \frac{\partial T}{\partial q_j} = 0$

ASSUME V INDEPENDENT OF ALL VELOCITIES $\{\dot{q}_i\}$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = \frac{d}{dt} P_j = \dot{P}_j = -\frac{\partial V}{\partial q_j} \equiv Q_j$$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$



$$\frac{\partial \vec{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\vec{r}_i^{\,tr} - \vec{r}_i}{dq_j} = \hat{n}$$

$$\Rightarrow Q_j = \sum_i \vec{F}_i \cdot \hat{n} = \hat{n} \cdot \vec{F}_{total}$$

$$P_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \hat{n} \cdot \sum_i m_i \dot{\vec{r}}_i$$

If V independent of q_j then

$$-\frac{\partial V}{\partial q_j} = Q_j = 0$$

$$\dot{P}_j = Q_j = 0$$

P_j IS CONSERVED

Energy function and E cons. for systems with $V \neq V(t)$

$$\mathcal{L} = \mathcal{L}(\{\dot{q}_i\}, \{q_i\}, \cancel{t})$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$

$$\frac{dL}{dt} = \underbrace{\sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right)\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot q_j}\frac{d\dot q_j}{dt}} + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \underbrace{\sum_j \frac{d}{dt}\left(\dot q_j \frac{\partial L}{\partial \dot q_j}\right)} + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\underbrace{\left(\sum_j \dot q_j \frac{\partial L}{\partial \dot q_j} - L\right)}_{h(\{\dot q_i\},\{q_i\},t)} + \frac{\partial L}{\partial t} = 0$$

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}$$

IF $\dfrac{\partial L}{\partial t} = 0$ then h CONSERVED

$$h(\{q_i\},\{\dot q_i\},t) = \underline{\sum_j p_j \dot q_j} - L(\{q_i\},\{\dot q_i\},t)$$

$$2T - (T - V) = T + V$$

TRIVIAL EXAMPLE

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x,y,z)$$

$$h = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}$$

$$= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V\right]$$

$$= T + V$$

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h = T + V ASSUMES

$$\mathcal{L} = \mathcal{L}_0(q,t) + \mathcal{L}_1(q,t)\dot{q} + \mathcal{L}_2(q,t)\dot{q}^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} = 2\mathcal{L}_2\dot{q}_2 + \mathcal{L}_1\dot{q}$$

$$h = \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L} = 2\mathcal{L}_2\dot{q}^2 + \mathcal{L}_1\dot{q} - \mathcal{L}_2\dot{q}^2 - \mathcal{L}_1\dot{q} - \mathcal{L}_0 = \overbrace{\mathcal{L}_2\dot{q}^2}^{T}\ \overbrace{-\mathcal{L}_0}^{V}$$

COUNTER EXAMPLE HW1 #2

ROTATING COORDINATES

$$q_1 = x\cos\omega t + y\sin\omega t$$

$$q_2 = -x\sin\omega t + y\cos\omega t$$

$$T = \frac{1}{2}m\left[(\dot q_1 - \omega q_2)^2 + (\dot q_2 + \omega q_1)^2\right]$$

$$P_1 = \frac{\partial \mathcal{L}}{\partial \dot q_1} = m(\dot q_1 - \omega q_2) \qquad V = 0$$

$$P_2 = \frac{\partial \mathcal{L}}{\partial \dot q_2} = m(\dot q_2 + \omega q_1)$$

$$h = P_1 \dot q_1 + P_2 \dot q_2 - \mathcal{L}$$

$$= m(\dot q_1^2 - \omega q_2 \dot q_1 + \dot q_2^2 + \omega q_1 \dot q_2) - T$$

$$\neq T$$

T CONTAINS TERMS 0^{th} + 1^{st} ORDER IN $\dot q$

EXAMPLE 2 PARTICLES IN A RELATIVE POTENTIAL

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad \vec{r} = \vec{r}_2 - \vec{r}_1$$

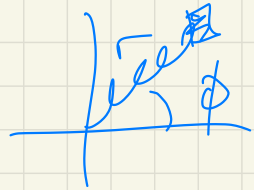
$$\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - V(\vec{r})$$

$$P_x = M\dot{X} \qquad P_y = M\dot{Y} \qquad P_z = M\dot{Z}$$

CONSERVED

$h = T + V = \text{Energy}$ ✓ conserved

MASS ON A CENTRAL SPRING



$$\mathcal{L} = \frac{1}{2}m\left(\dot r^2 + r^2\dot\phi^2\right) - \frac{k}{2}(r-L)^2$$

ϕ ignorable $\Rightarrow$ $p_\phi = mr^2\dot\phi =$ constant

cyclic

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot r} = m\dot r$$

$$E = h = p_r\dot r + p_\phi\dot\phi - \mathcal{L}$$

$$= \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\dot\phi^2 + \frac{k}{2}(r-L)^2$$

conserved

$$E = \underbrace{\frac{1}{2}m\dot r^2}_{T_{eff}} + \underbrace{\frac{p_\phi^2}{2mr^2} + \frac{k}{2}(r-L)^2}_{V_{eff}}$$

$$m\ddot r = -\frac{\partial V^{eff}}{\partial r} = \frac{p_\phi^2}{mr^3} - k(r-L)$$

$$ma = F$$

$$m\ddot{r} = \frac{p_\phi^2}{mr^3} - k(r-L)$$

STATIONARY $\ddot{r} = 0$

$$\frac{p_\phi^2}{mr^3} = k(r-L) \Rightarrow r = r(m, L, k, p_\phi)$$

EXPAND AROUND STATIONARY SOLUTION

$$V^{eff}(r) = V^{eff}(r_0) + \left.\frac{\partial V^{eff}}{\partial r}\right|_{r_0} (r - r_0) \quad 0$$

$$+ \left.\frac{\partial^2 V^{eff}}{\partial r^2}\right|_{r_0} (r - r_0)^2$$

$\Rightarrow$ small oscillation frequency

$$\omega = \left[\left.\frac{\partial^2 V^{eff}}{\partial r^2}\right|_{r_0} \Big/ m \right]^{1/2}$$