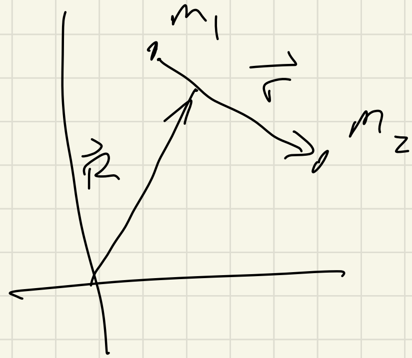
2 body $\rightarrow$ 1 body



$$U = U(\{\vec{r}\}, \{\dot{\vec{r}}\}, \dots)$$

$$U \neq U(\vec{R})$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$\vec{R}$ is cyclic

$\vec{P}$ conserved

CHOOSE A REFERENCE FRAME WHERE $\vec{P} = M\dot{\vec{R}} = 0$

CONSERVATIVE CENTRAL FORCE

$$V = V(r) \qquad \vec{F} = -\frac{\partial V}{\partial r}\hat{r}$$

SPHERICALLY SYMMETRIC

ANY ROTATION LEAVES THE E.O.M.
UNCHANGED $\Rightarrow$ CONSERVED
QUANTITIES

$\vec{L} = \vec{r} \times \vec{p}$ CONSERVED

3 CONSTRAINTS

$\Rightarrow \vec{r}$ IN A PLANE $\perp$ to $\vec{L}$

SPECIAL CASE $L = 0$

$\vec{r} \parallel \vec{p}$ STRAIGHT LINE MOTION

CHOOSE $\hat{z} \parallel \vec{L} \Rightarrow \phi = \pi/2$

(2 CONSTRAINTS)

NOTATION

(r, θ, ϕ) $\quad z = r\cos\theta$

$x = r\sin\theta\cos\phi$

(r, ϕ, θ) $\quad z = r\cos\phi$

$x = r\sin\phi\cos\theta$

ALL MOTION IS IN X-Y PLANE

$$\mathcal{L} = T - V$$
$$= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

θ CYCLIC

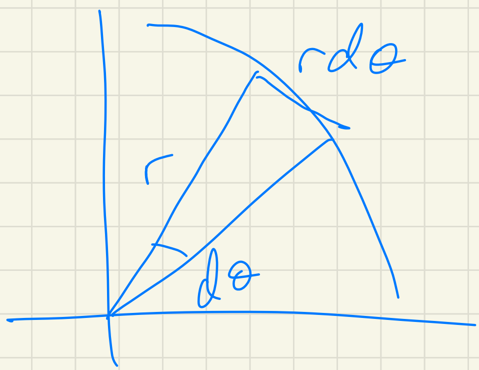
$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} \equiv l_z$$

ALTERNATIVELY $V = V(r)$ CENTRAL

$\Rightarrow$ MOTION IS PLANAR

AREA SWEPT

$dA = r^2 d\theta \frac{1}{2}$



AREAL VELOCITY

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

CONSERVATION OF $p_\theta \Leftrightarrow$ CONSTANT AREAL VELOCITY

KEPLER'S 2ND LAW

$$P_\theta = \mu r^2 \dot{\theta} = \mu |\vec{r} \times \dot{\vec{r}}| = l_z$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = \mu \ddot{r} - \mu r \dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$\mu \ddot{r} - \mu r \dot{\theta}^2 = f(r)$$

FORCE

$$\mu \ddot{r} - \frac{l_z^2}{\mu r^3} = f(r)$$

---

(1) $V \neq V(t) \Rightarrow E$ conserved

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

---

(2) $$\mu \ddot{r} = \frac{l_z^2}{\mu r^3} + f(r)$$

$$= -\frac{d}{dr}\left(V + \frac{1}{2}\frac{l_z^2}{\mu r^2}\right)$$

multiply both sides by $\dot{r}$

LHS: $\mu \ddot{r}\dot{r} = \frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2\right)$

RHS: $-\frac{d}{dr}(\quad)\frac{dr}{dt} = -\frac{d}{dt}(\quad)$

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{d}{dt}\left(V + \frac{1}{2}\frac{l_z^2}{\mu r^2}\right)$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l_z^2}{\mu r^2} + V\right) = 0$$

E CONSERVED

---

③ $h = \sum_i p_i \dot{q}_i - \mathcal{L}$

$$= (\mu\dot{r})\dot{r} + \mu r^2\dot{\theta}^2 - \left[\frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V\right]$$

$$E = h = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V$$

Solve $\dot{r} = \left[\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)\right]^{1/2}$

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{}} \Rightarrow t(r)$$

invert to get $r(t)$

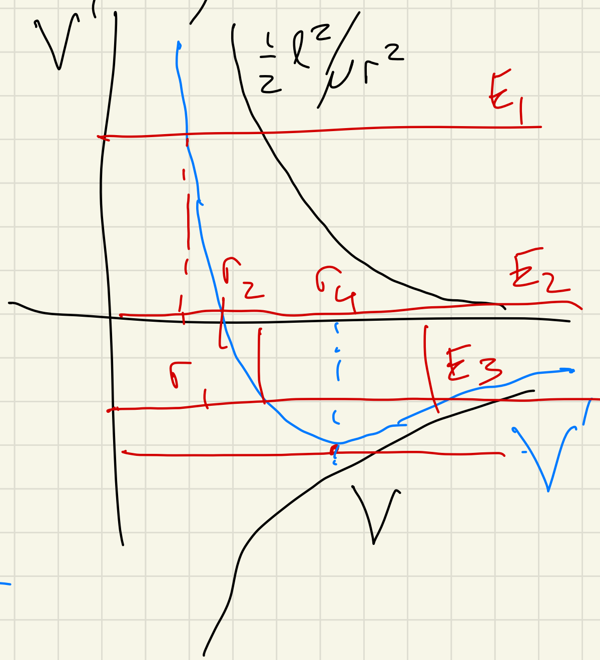
Solved!

Orbits

$$f' = f + \frac{l_z^2}{\mu r^3}$$

$$V' = V + \frac{1}{2}\frac{l^2}{\mu r^2}$$

$$E = V' + \frac{1}{2} m \dot{r}^2$$



$|V|$ decreases slower than $1/r^2$ as $r \to \infty$

$|V|$ increases slower than $1/r^2$ as $r \to 0$

① $E_1 > 0$ $\quad r > r_1$

$\frac{1}{2}\mu \dot{r}^2 = E_1$ @ $r \to \infty$

(If $V \sim -k/r$ hyperbola)

② $E_2 = 0$ $\quad r > r_2$

$\frac{1}{2}\mu \dot{r}_\infty^2 = 0$

If $V \sim -\frac{k}{r}$ parabola

③ $E_3 < 0$ $\quad r_3^{min} < r < r_3^{max}$

ORBIT BOUNDED

CLOSED?

If $V \sim -\frac{k}{r}$, ellipse

④ $E_4 = V'_{min}$ $\quad r = r_4$

CIRCLE

$V(r) = -\frac{a}{r^3}$ FAILS THE LIMITS

① $E_1 > V'$ $\forall r$

r unbounded

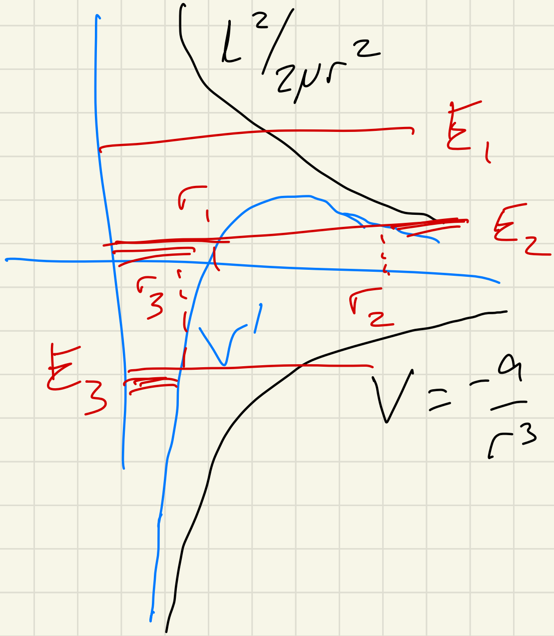
② $E_2 > 0$

EITHER $r < r_1$

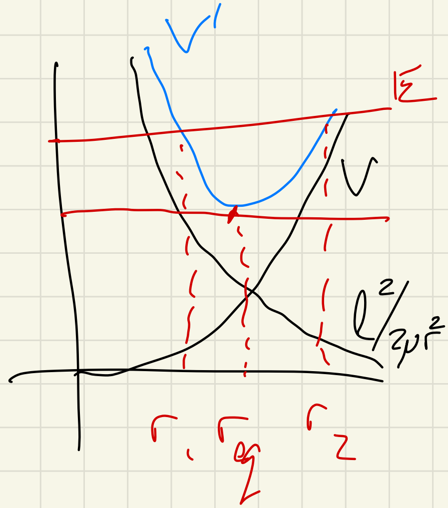
OR $r > r_2$

③ $E_3 < 0$

$r < r_3$



ISOTROPIC HARMONIC OSCILLATOR



$f = -kr \quad V = \frac{1}{2}kr^2$

a) $\ell = 0 \implies$ 1D H.O.

$V' = V$

b) $\ell \neq 0$

$$V' = \frac{\ell^2}{2\mu r^2} + \frac{k}{2}r^2$$

EQUILIBRIUM $\quad 0 = \frac{\partial V'}{\partial r} = \frac{-\ell^2}{\mu r^3} + kr$

$$r_{eq}^4 = \frac{\ell^2}{\mu k}$$

$$V'_{eq} = \frac{\ell^2}{2\mu \ell/\sqrt{\mu k}} + \frac{k}{2}\frac{\ell}{\sqrt{\mu k}} = \ell\sqrt{\frac{k}{\mu}} = \ell\omega$$

$$T'_{eq} = \frac{1}{2}m\dot{r}^2 = 0$$

PERIOD (USE AREA)

$$\left[\ell = \sqrt{\mu k}\, r_{eq}^2 \right]$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{\ell}{2\mu}$$

$$\text{PERIOD} = \frac{\text{AREA}}{\text{RATE}} = \frac{A}{dA/dt} = \frac{\pi r_{eq}^2}{\ell / 2\mu}$$

$$= 2\pi \mu r_{eq}^2 / \ell$$

$$= \frac{2\pi \mu r_{eq}^2}{\sqrt{\mu k}\, r_{eq}^2} = 2\pi \sqrt{\frac{\mu}{k}} = \frac{2\pi}{\omega}$$

CARTESIAN $\mathcal{L} = \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2) + \frac{k}{2}(x^2 + y^2)$

$$x = A_x \cos(\omega t + \phi_x) \quad y = A_y \sin(\omega t + \phi_y)$$

AT EQUIL: $A_x = A_y \quad \phi_x = \phi_y$

SMALL OSCILLATIONS AROUND r_{eq}

$$V'(r) = V'(r_{eq}) + \cancelto{0}{\frac{\partial V'}{\partial r}} \delta r + \frac{1}{2} \frac{\partial^2 V'}{\partial r^2}\bigg|_{r_{eq}} \delta r^2$$

$$= C + \frac{1}{2}\left(\frac{3\ell^2}{\mu r_{eq}^4} + k\right)\delta r^2$$

$\delta r = r - r_{eq}$

$$= C + \frac{1}{2}(4k)(\delta r)^2$$

$\ell^2 = \mu k r_{eq}^4$

$$\omega_{osc} = \sqrt{\frac{k'}{\mu}} = \sqrt{\frac{4k}{\mu}} = 2\omega$$

CLOSED ORBIT

BECAUSE 2 IS

RATIONAL

TURNING POINTS (MIN, MAX r)

$$E = \frac{\ell_z^2}{2\mu r^2} + \frac{k}{2} r^2 + \cancel{\frac{1}{2}\mu \dot{r}^2} = 0$$

$\dot{r} = 0$ FOR EXTREME r

$$\frac{k}{2} r^4 - E r^2 + \frac{\ell^2}{2\mu} = 0$$

$$r^4 - \frac{2E}{k} r^2 + \frac{\ell^2}{\mu k} = 0$$

$$r^2_{\substack{max \\ min}} = \frac{\left(\frac{\cancel{2}E}{k} \pm \left[\frac{\cancel{4}E^2}{k^2} - \frac{\cancel{4}\ell^2}{\mu^2 k^2} \right]^{1/2} \right)}{\cancel{2}}$$