VIRIAL THM

$$G = \sum_i \vec{p}_i \cdot \vec{r}_i$$

$$\frac{dG}{dt} = \sum_i \underbrace{\dot{\vec{p}}_i \cdot \vec{r}_i}_{\vec{F}_i \cdot \vec{r}_i} + \sum_i \underbrace{\vec{p}_i \cdot \dot{\vec{r}}_i}_{2T = \sum m v^2}$$

$$\frac{d}{dt}\left(\sum_i \vec{p}_i \cdot \vec{r}_i\right) = 2T + \sum_i \vec{F}_i \cdot \vec{r}_i$$

TIME AVERAGE

$$\frac{1}{\tau}\int_0^\tau \left(\frac{dG}{dt}\right) dt = \overline{\frac{dG}{dt}} = \overline{2T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i}$$

$$\overline{2T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i} = \frac{1}{\tau}\left[\cancelto{0}{G(\tau) - G(0)}\right]$$

IF MOTION IS PERIODIC OR IF SYSTEM IS BOUNDED THEN

$\Delta G = $ FINITE, $\tau \to \infty$

$$\overline{T} = -\frac{1}{2}\overline{\sum_i \vec{F}_i \cdot \vec{r}_i}$$

VIRIAL THM

IF $\vec{F} = -\vec{\nabla} V(\vec{r})$: $\overline{T} = \frac{1}{2} \overline{\sum_i \vec{\nabla} V \cdot \vec{r}}$

CASE: SINGLE PARTICLE IN CENTRAL $V = V(r)$

$$\overline{T} = \frac{1}{2} \overline{\frac{\partial V}{\partial r} r}$$

IF $V = a r^{n+1}$: $\frac{\partial V}{\partial r} r = (n+1) V$

HARMONIC OSCILLATOR $n = 1$

$$\overline{T} = \frac{1}{2} \overline{V}$$

GRAVITY $n = -2$ $\overline{T} = -\frac{1}{2} \overline{V}$

ESCAPE VELOCITY $= \sqrt{2}$ CIRCULAR VELOCITY

DETERMINE MASSES OF GALAXIES IN CLUSTERS

ORBIT EQ (use m, not μ)

FIND $r(\theta)$ 1st, $r(t), \theta(t)$ later

$$l = mr^2 \frac{d\theta}{dt} \qquad \frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}$$

$$m\frac{d^2r}{dt^2} - \frac{l}{mr^3} = f(r)$$

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l}{mr^3} = f(r)$$

$$u = 1/r \qquad f = -\frac{\partial V}{\partial r} = -\frac{dV}{dr}$$

$$\frac{du}{dr} = -\frac{1}{r^2} = -u^2 \qquad f = -\frac{dV}{du}\frac{du}{dr} = u^2\frac{dV}{du}$$

$$\frac{l^2}{m}u^2\frac{d}{d\theta}\left(u^2\frac{d\,1/u}{d\theta}\right) = \frac{l^2}{m}u^2\frac{d}{d\theta}\left(u^2\left(-u^{-2}\frac{du}{d\theta}\right)\right)$$

$$= -\frac{l^2u^2}{m}\frac{d^2u}{d\theta^2}$$

$$-\frac{l^2 \cancel{u^2}}{m}\frac{d^2u}{d\theta^2} - \frac{l^2 u^{\cancel{3}}}{m} = \cancel{u^2}\,\frac{dV}{du}$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right)$$

SYMMETRIC FOR $\pm\theta$

CHOOSE $\theta = 0$ AT A TURNING POINT
(r IS MAX OR MIN)

THEN $u_0 = u(0)$

$$\left.\frac{du}{d\theta}\right|_0 = 0$$

$$dt = \frac{dr}{\left[\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)\right]^{1/2}}$$

$$d\theta = \frac{l}{mr^2}\,dt$$

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\left[\frac{2m}{l^2}(E - V) - u^2\right]^{1/2}}$$

$V = ar^{n+1}$ $\quad n = 1, -2, -3$ TRIG FNS

HO, G,

$n = 5, 3, 0 \ldots 7$ ELLIPTICAL FNS

CLOSED ORBITS?

SMALL PERTURBATIONS

$$U = U_0 + a\cos(\beta\theta)$$

$\Rightarrow$ CONSTRAINT ON β

β RATIONAL $\Rightarrow$ CLOSED

BIG PERTURBATIONS

ONLY $V = ar^{n+1}$ $\quad n = 1$ HO

$n = -2$ G

CLOSED

GRAVITY $f = -k/r^2$ $V = -k/r$

$$\theta = \theta' - \int_{u_0}^{u} \frac{du}{\left[\frac{2mE}{l^2} + \frac{2mk}{l} u - u^2\right]^{1/2}}$$

look it up

$$\int \frac{dx}{(\alpha + \beta x + \gamma x^2)^{1/2}} = \frac{1}{\sqrt{-\gamma}} \cos^{-1}\left(\frac{-\beta + 2\gamma x}{\sqrt{q}}\right)$$

$$q = \beta^2 - 4\alpha\gamma$$

$$\alpha = \frac{2mE}{l^2} \qquad \beta = \frac{2mk}{l^2} \qquad \gamma = -1$$

$$q = \left(\frac{2mk}{l^2}\right)^2 \left(1 + \frac{2El^2}{mk^2}\right)$$

$$\theta = \theta' - \cos^{-1}\left[\frac{l^2 u/mk - 1}{\left[1 + \frac{2El^2}{mk^2}\right]^{1/2}}\right]$$

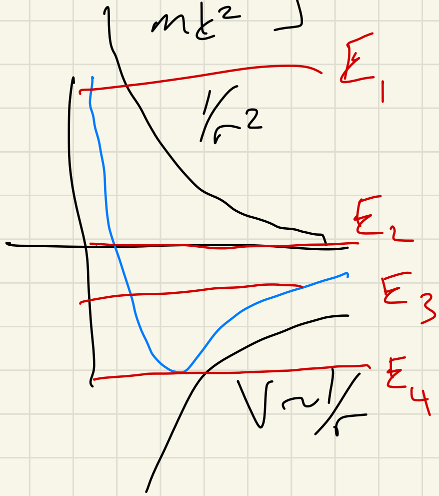
$$\frac{1}{r} = u = \frac{mk}{l^2}\left(1 + \left[1 + \frac{2El^2}{mk^2}\right]^{1/2} \cos(\theta - \theta')\right)$$

θ' = TURNING ANGLE

CONIC WITH ONE FOCUS @ ORIGIN

$$\frac{1}{r} = C\left[1 + e\cos(\theta - \theta')\right]$$

ECCENTRICITY $e = \left[1 + \frac{2El^2}{mk^2}\right]^{1/2}$



$E > 0 \Rightarrow e > 1$ HYPERBOLA

$E = 0 \Rightarrow e = 1$ PARABOLA

$E < 0 \Rightarrow e < 1$ ELLIPSE

$E = -\frac{mk^2}{2l^2} \Rightarrow e = 0$ CIRCLE

CIRCULAR ORBIT $E = T + V$

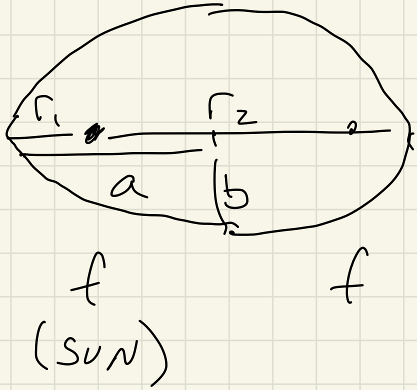
VIRIAL THm: $= -\frac{V}{2} + V = \frac{V}{2}$

$E = -\frac{k}{2r_0}$ $\qquad \frac{k}{r_0^2} = \frac{l^2}{mr_0^3}$ $\qquad r_0 = \frac{l^2}{mk}$

$E = -\frac{mk^2}{2l^2}$

a = semi major axis

$= \frac{1}{2}(r_1 + r_2)$



@ $r_1, r_2 : \dot{r} = 0$

$$E = \frac{l^2}{2mr^2} - \frac{k}{r}$$

r_1 = perihelion

r_2 = aphelion

$$r^2 + \frac{k}{E} r - \frac{l^2}{2mE} = 0$$

$k, l \geq 0$

$E < 0$

$$r_{1,2} = \frac{-\frac{k}{E} \pm \sqrt{\frac{k^2}{E^2} + 4l^2/2mE}}{2}$$

$$\frac{1}{2}(r_1 + r_2) = -\frac{k}{2E} = a$$

INDEPENDENT OF l

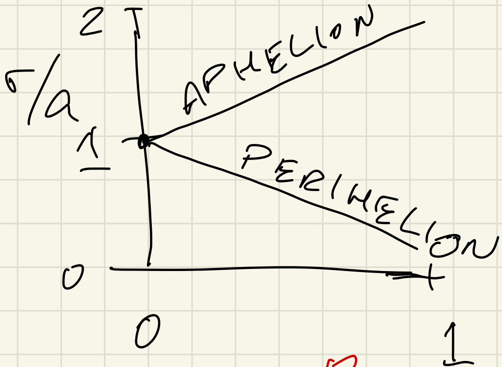
$$e = \left[1 - l^2/mka\right]^{1/2}$$

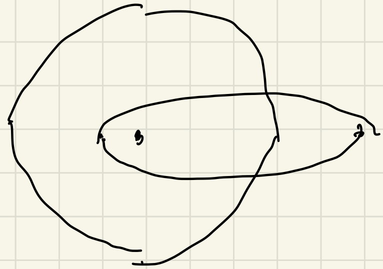
FOR $0 < e < 1$

$$r = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta')}$$

$\cos\theta = 1$ ~~$\theta = 0$~~ $r = a(1-e)$

-1 $r = a(1+e)$





$$\vec{V} = \cancel{V_r \hat{r}}^{\,0} + V_\theta \hat{\theta}$$

@ TURNING POINTS

V_θ MAX AT PERIHELION

V_θ MIN AT APHELION

TIME

$$t = \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr}{\left[\frac{k}{r} - \frac{l^2}{2mr^2} + E\right]^{1/2}}$$

$$= \frac{l^3}{mk^2} \int_{\theta_0}^{\theta} \frac{d\theta}{\left[1 + e\cos(\theta - \theta')\right]^2}$$

CAN GET $t(r)$, $t(\theta)$. UGLY

INVERT TO GET $r(t)$, $\theta(t)$

REALLY UGLY

CASE $e = 1$ PARABOLA

(CHOOSE $\theta' = 0$)

$$1 + \cos\theta = \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$t = \frac{\ell^3}{4mk^2}\int_0^\theta \frac{1}{\cos^4\frac{\theta}{2}}\,d\theta$$

$$\sec^4\theta = \frac{1}{\cos^4\theta}$$

$$x = \tan\theta/2$$

$$dx = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$$

$$t = \frac{\ell^3}{4mk^2}\int_0^{\tan\theta/2}(1+x^2)\,dx$$

$$1+\tan^2\theta = \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}$$

$$t = \frac{\ell^3}{2mk^2}\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\theta/2\right)$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\infty < t < +\infty$$

CASE $e < 1$ ELLIPSE

DEFINE ψ: $r = a(1 - e\cos\psi)$

PERIHELION $= \dfrac{a(1-e^2)}{1+e\cos\theta}$

$\psi = \theta = 0$

APHELION

$\psi = \theta = \pi$

$$t = \sqrt{\frac{ma^3}{k}} \int_0^{\psi} (1 - e\cos\psi)\, d\psi$$

FOR $\psi = 2\pi$ $\quad \tau = 2\pi a^{3/2} \sqrt{\dfrac{m}{k}} \leftarrow \mu$

$$\tau^2 \propto a^3$$

$k = G M_\odot m \qquad \mu = \dfrac{M_\odot m}{M_\odot + m}$

$\dfrac{\mu}{k} = \dfrac{M_\odot + m}{G} \qquad \dfrac{M_{JUPITER}}{M_\odot} = 10^{-3}$