CONSERVATIVE SYSTEMS $V = V(\{q_i\})$

TIME INDEPENDENT

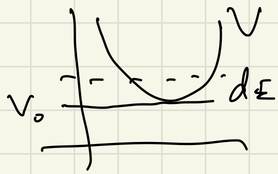
EQUIL $Q_i = -\left.\frac{\partial V}{\partial q_i}\right|_0 = 0$

FORCES VANISH

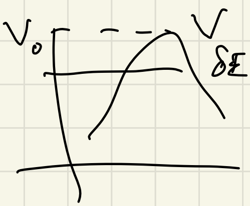
V EXTREMUM

IF $\{\dot{q}_i\} = 0 \ \forall i$ THEN SYSTEM WILL STAY IN EQUIL

FOCUS ON STABLE CASES



STABLE



UNSTABLE

$q_i = q_{0i} + \eta_i$

$$V(\{q_i\}) = \underbrace{V(\{q_{0i}\})}_{=V_0} + \cancel{\left.\frac{\partial V}{\partial q_i}\right|_0}^{\,0} \eta_i + \frac{1}{2}\left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0 \eta_i \eta_j$$

REPEATED INDICES $\Rightarrow$ IMPLIED SUMMATION

$$V(\{\eta_i\}) = \frac{1}{2}\left.\frac{\partial^2 V}{\partial q_i q_j}\right|_0 \eta_i \eta_j = \frac{1}{2} V_{ij} \eta_i \eta_j$$

$V_{ij} = 0$ IF V INDEPENDENT OF q_i

V_{ij} symmetric

$$T = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} m_{ij} \dot{\eta}_i \dot{\eta}_j$$

T HOMOGENOUS QUADRATIC

IF $m_{ij}(\{q_i\})$ THEN WE CAN EXPAND IT

BUT T IS ALREADY 2ND ORDER IN $\dot{\eta}$

$$T = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j \qquad T_{ij} = m_{ij}\big|_0$$

T_{ij} MUST BE SYMMETRIC

$$\mathcal{L} = \frac{1}{2}\left(T_{ij}\dot{\eta}_i\dot{\eta}_j - V_{ij}\eta_i\eta_j\right) \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}_i} - \frac{\partial \mathcal{L}}{\partial \eta_i} = 0$$

EQ OF MOTION $\quad T_{ij}\ddot{\eta}_j + V_{ij}\eta_j = 0$

IMPLIED $\sum_j$

MOST PROBLEMS DIAGONAL IN T

$$\mathcal{L} = \frac{1}{2}\left(T_i\dot{\eta}_i^2 - V_{ij}\eta_i\eta_j\right)$$

$$\underbrace{T_i\ddot{\eta}_i}_{\text{NO SUM OVER } i} + \sum_j V_{ij}\eta_j = 0$$

$$T = \frac{1}{2}\sum_{ij} T_{ij}\dot{\eta}_i\dot{\eta}_j$$

$$= \frac{1}{2}\left(\dot{\eta}_1, \ldots, \dot{\eta}_n\right)\begin{pmatrix} & & \\ & T_{ij} & \\ & & \end{pmatrix}\begin{pmatrix} \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_n \end{pmatrix}$$

$$= \frac{1}{2}\left(\dot{\vec{\eta}}\right)^T \overleftrightarrow{T}\,\dot{\vec{\eta}}_i$$

$$V = \frac{1}{2}\sum_{ij} V_{ij}\eta_i\eta_j$$

$$= \frac{1}{2}\left(\eta_1, \ldots, \eta_n\right)\begin{pmatrix} V_{ij} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} = \frac{1}{2}\left(\vec{\eta}\right)^T \overleftrightarrow{V}\,\vec{\eta}$$

EOM $\quad T_{ij}\ddot{\eta}_j + V_{ij}\eta_j = 0$

$$\begin{pmatrix} T_{ij} \end{pmatrix}\begin{pmatrix} \ddot{\eta}_1 \\ \vdots \\ \ddot{\eta}_n \end{pmatrix} + \begin{pmatrix} V_{ij} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} = 0$$

$$\overleftrightarrow{T}\,\ddot{\vec{\eta}} + \overleftrightarrow{V}\,\vec{\eta} = 0$$

OSCILLATORY ANSATZ $\eta_i = C a_i e^{-i\omega t}$

$$-\omega^2 \overleftrightarrow{\mathbb{T}}\, \vec{a} + \overleftrightarrow{\mathbb{V}}\, \vec{a} = \vec{0} \qquad C a_i \in \mathbb{C}$$

$$(\overleftrightarrow{\mathbb{V}} - \lambda \overleftrightarrow{\mathbb{T}})\,\vec{a} = 0 \qquad \lambda = \omega^2 \qquad \vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad \vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

LINEAR HOMOGENOUS

NONTRIVIAL SOLUTION IFF $\det(\overleftrightarrow{\mathbb{V}} - \lambda \overleftrightarrow{\mathbb{T}}) = 0$

$$\begin{vmatrix} V_{11} - \lambda T_{11} & V_{12} - \lambda T_{12} & \cdots \\ V_{21} - \lambda T_{21} & \ddots & \\ V_{31} - \lambda T_{31} & & \ddots \\ \vdots & & \end{vmatrix} = 0$$

FOR EACH λ_k THAT IS A SOLUTION

$\exists$ EIGENVECTOR $\vec{a}$ $\qquad \lambda_k \to \vec{a}_k \quad k = 1, \ldots, n$

USUAL $\overleftrightarrow{\mathbb{H}}\, \vec{\psi} = E\, \vec{\psi}$

$$\overleftrightarrow{\mathbb{V}}\, \vec{a} = \lambda \overleftrightarrow{\mathbb{T}}\, \vec{a}$$

SHOW THAT $\lambda_k \in \mathbb{R}$ $\qquad \lambda_k > 0$

$\{\vec{a}_k\} = \{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_n\}$ ORTHOGONAL

REAL

$\overleftrightarrow{\mathbb{V}}, \overleftrightarrow{\mathbb{T}} \in \mathbb{R}$

$$\overleftrightarrow{\mathbb{V}}\, \vec{a}_k = \lambda_k \overleftrightarrow{\mathbb{T}}\, \vec{a}_k$$

$$\vec{a}_\ell^{\,\dagger}\, \overleftrightarrow{\mathbb{V}} = \lambda_\ell^* \,\vec{a}_\ell^{\,\dagger}\, \overleftrightarrow{\mathbb{T}}$$

$$\vec{a}_\ell^{\,\dagger}\, \overleftrightarrow{\mathbb{V}}\, \vec{a}_k = \lambda_k \,\vec{a}_\ell^{\,\dagger}\, \overleftrightarrow{\mathbb{T}}\, \vec{a}_k$$

$$\vec{a}_\ell^{\,\dagger}\, \overleftrightarrow{\mathbb{V}}\, \vec{a}_k = \lambda_\ell^* \,\vec{a}_\ell^{\,\dagger}\, \overleftrightarrow{\mathbb{T}}\, \vec{a}_k$$

$$0 = (\lambda_k - \lambda_\ell^*)\, \vec{a}_\ell^{\,\dagger}\, \overleftrightarrow{\mathbb{T}}\, \vec{a}_k \;\overset{k=\ell}{\Longrightarrow}\; (\lambda_k - \lambda_k^*)\, \vec{a}_k^{\,\dagger}\, \overleftrightarrow{\mathbb{T}}\, \vec{a}_k = 0$$

$$\vec{a}_k = \vec{\alpha}_k + i\vec{\beta}_k \qquad \vec{a}_k^{\dagger} = \vec{\alpha}_k^T - i\vec{\beta}_k^T$$

$$\underbrace{\vec{a}_k^{\dagger}\overleftrightarrow{\mathbb{T}}\vec{a}_k}_{\mathbb{R}} = \vec{\alpha}_k^T\overleftrightarrow{\mathbb{T}}\vec{\alpha}_k + \vec{\beta}_k^T\overleftrightarrow{\mathbb{T}}\vec{\beta}_k + \cancel{i\left(\vec{\alpha}_k^T\overleftrightarrow{\mathbb{T}}\vec{\beta}_k - \vec{\beta}_k^T\overleftrightarrow{\mathbb{T}}\vec{\alpha}_k\right)} \rightarrow 0$$

$\overleftrightarrow{\mathbb{T}}$ SYMMETRIC

$\alpha\mathbb{T}\alpha$ and $\beta\mathbb{T}\beta$ ARE KINETIC ENERGY-LIKE

$$\Rightarrow \lambda_k \in \mathbb{R} \qquad \vec{a}_k \in \mathbb{R}$$

SOLVE $\overleftrightarrow{\mathbb{V}}\vec{a}_k = \lambda_k \overleftrightarrow{\mathbb{T}}\vec{a}_k$ MULTIPLY BY $\vec{a}_k^T$

$$\lambda_k = \frac{\vec{a}_k^T\overleftrightarrow{\mathbb{V}}\vec{a}_k \text{ SCALAR}}{\vec{a}_k^T\overleftrightarrow{\mathbb{T}}\vec{a}_k \text{ SCALAR}} \qquad \lambda_k \cancel{\overset{?}{=}} \frac{\overleftrightarrow{\mathbb{V}}\vec{a}_k \text{ VECTOR}}{\overleftrightarrow{\mathbb{T}}\vec{a}_k \text{ VECTOR}}$$

$\mathbb{V}$ IS MINIMUM @ EQUILIBRIUM

STABILITY $\Rightarrow \vec{a}_k^T\mathbb{V}\vec{a}_k \geq 0$

BOTTOM KE-like $\Rightarrow \vec{a}_k\mathbb{T}\vec{a}_k > 0$

$$\lambda_k \geq 0 \;\; \forall k \Rightarrow \omega_k = \sqrt{\lambda_k} \in \mathbb{R} \text{ REAL}$$

$$\Rightarrow \vec{a}_k \in \mathbb{R}$$

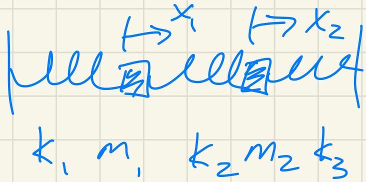
NORMAL MODES $C\vec{a}_k e^{i\sqrt{\lambda_k}t}$

$$\left(\eta_i = Ca_i e^{-i\omega t}\right) \Rightarrow \{q_i\} = \{q_{i0} + \eta_i\}$$

$$= \left\{q_{i0} + C\left(\vec{a}_k\right)_i e^{i\sqrt{\lambda_k}t}\right\}$$

GENERAL $\vec{\eta}(t) = \mathrm{Re}\left(\sum_k C_k \vec{a}_k e^{i\sqrt{\lambda_k}t}\right)$

EXAMPLE



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_1 - x_2)^2 + \frac{1}{2} k_3 x_2^2$$

$$= \frac{1}{2}\left[(k_1 + k_2) x_1^2 + (k_2 + k_3) x_2^2 - 2k_2 x_1 x_2\right]$$

$$\overleftrightarrow{T} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad \overleftrightarrow{V} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}$$

SIMPLIFY $m_1 = m_2$ $\quad k_1 = k_3 = k$

$$\left|\overleftrightarrow{V} - \lambda \overleftrightarrow{T}\right| = \begin{vmatrix} k + k_2 - \lambda m & -k_2 \\ -k_2 & k + k_2 - \lambda m \end{vmatrix}$$

$$= \left((k + k_2)^2 + \lambda^2 m^2 - 2(k + k_2)\lambda m\right) - k_2^2 = 0$$

$$= \lambda^2 - \frac{2(k + k_2)}{m}\lambda + \frac{k^2 + 2k k_2}{m^2} = 0$$

$$\lambda = \frac{\dfrac{2(k + k_2)}{m} \pm \left[\dfrac{4(\cancel{k^2} + \cancel{2k k_2} + k_2^2)}{m^2} - \dfrac{\cancel{4k^2} + \cancel{8k k_2}}{m^2}\right]^{1/2}}{2}$$

$$\lambda = \frac{k + k_2}{m} \pm \frac{k_2}{m} \qquad \lambda_1 = \frac{k + 2k_2}{m} \qquad \lambda_2 = \frac{k}{m}$$

$$\left(\overleftrightarrow{V} - \lambda_1 \overleftrightarrow{T}\right)\vec{a}_1 = \begin{pmatrix} k + k_2 - (k + 2k_2) & -k_2 \\ -k_2 & k + k_2 - (k + 2k_2) \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$= \begin{pmatrix} -k_2 & -k_2 \\ -k_2 & -k_2 \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0 \qquad \vec{a}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = \frac{k}{m}$$

$$\left(\overleftrightarrow{V} - \lambda_2 \overleftrightarrow{T}\right)\vec{a}_2 = \begin{pmatrix} k+k_2-k & -k_2 \\ -k_2 & k+k_2-k \end{pmatrix}\begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}\begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0 \implies \vec{a}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{k+2k_2}{m}} \qquad \omega_2 = \sqrt{\lambda_2} = \sqrt{k/m}$$

PUT IN INITIAL CONDITIONS

$$x_1(0) = C \qquad x_2(0) = 0 \qquad \dot{x}_1(0) = \dot{x}_2(0) = 0$$

$$\vec{\eta}(t) = C_1 \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\vec{a}_1} e^{i\omega_1 t} + C_2 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{a}_2} e^{i\omega_2 t}$$

$$x_1 : \mathrm{Re}(C_1 + C_2) = C \qquad \dot{x}_1 \quad \mathrm{Im}(C_1 + C_2) = 0$$

$$x_2 : \mathrm{Re}(C_1 - C_2) = 0 \qquad \dot{x}_2 \quad \mathrm{Im}(C_1 - C_2) = 0$$

$$C_1 = C_2 = \frac{C}{2}$$