DESCRIBE THE RIGID BODY

N PARTICLES $\Rightarrow$ 3N DOF

$\frac{N(N-1)}{2}$ CONSTRAINTS

CM $\vec{R}$: 3

ORIENTATION: 3

FIX $|\vec{r}_{12}|$, $|\vec{r}_{13}|$ . . .

PLACE 1^{st} PARTICLE : $\vec{r}_1$ (3)

" 2^{ND} " : 2

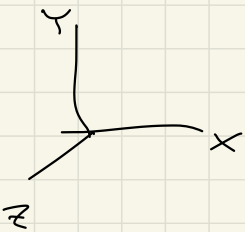
" 3^{RD} " : 1

$$\overline{6}$$

SPACE COORD $\hat{i}, \hat{j}, \hat{k}$

BODY " $\hat{i}', \hat{j}', \hat{k}'$

$\Rightarrow$ 9 COSINES



$$\cos\theta_{11} = \hat{i}' \cdot \hat{i}$$
$$\cos\theta_{12} = \hat{i}' \cdot \hat{j}$$
$$\theta_{13} = \hat{i}' \cdot \hat{k}$$

$$\cos\theta_{21} = \hat{j}' \cdot \hat{i}$$

$$\hat{i}' = \cos\theta_{11}\,\hat{i} + \cos\theta_{12}\,\hat{j} + \cos\theta_{13}\,\hat{k}$$
$$\hat{j}' = \cos\theta_{21}$$
$$k' = \cos\theta_{31}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$$

$$x' = \vec{r} \cdot \hat{i}' = \cos\theta_{11}\,x + \cos\theta_{12} + \cos\theta_{13}\,z$$

y'

z'

If $\hat{i}'$ $\hat{j}'$ $\hat{k}'$ FIXED WITH RESPECT TO THE BODY, THEN WILL CHANGE AS IT ROTATES

ONLY 3 θ_{ij} INDEPENDENT

$$\left.\begin{aligned}\hat{i}'\cdot\hat{i}' = \hat{j}'\cdot\hat{j}' = \hat{k}'\cdot\hat{k}' = 1\\ \hat{i}'\cdot\hat{j}' = \hat{j}'\cdot\hat{k}' = \hat{k}'\cdot\hat{i}' = 0\end{aligned}\right\}\ 6 \text{ CONSTRAINTS}$$

SWITCH NOTATION $xyz \to x_1 x_2 x_3$

$\cos\theta_{ij} \Rightarrow a_{ij}$

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

$$\vec{x}' = \overleftrightarrow{A}\vec{x} \qquad x_i' = a_{ij}x_j \text{ (implied sum)}$$

KNOW $\vec{x}'\cdot\vec{x}' = \vec{x}\cdot\vec{x}$ $\qquad \vec{x}'^T\vec{x}' = \vec{x}^T\underbrace{\overleftrightarrow{A}^T\overleftrightarrow{A}}_{=\overleftrightarrow{\mathbb{1}}}\vec{x}$

$$\vec{x}'^T = \vec{x}^T\overleftrightarrow{A}^T \qquad a_{ij}a_{ik} = \delta_{jk}$$

$$\overleftrightarrow{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

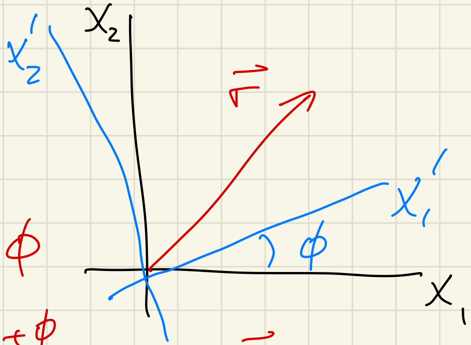
SPECIFIC ROTATION AROUND Z $\begin{pmatrix} a_{11} & a_{12} & 0\\ a_{21} & a_{22} & 0\\ 0 & 0 & 1\end{pmatrix}$

$$a_{ij}a_{ik} = \delta_{jk} \quad j,k = 1,2$$

12 AND 21 ARE THE SAME

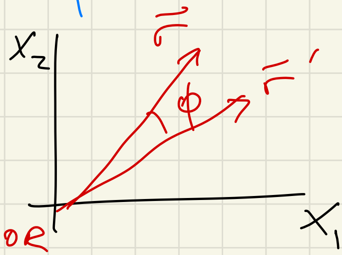
3 CONSTRAINTS $\Rightarrow$ 1 DOF

$$x_1' = x_1\cos\phi + x_2\sin\phi$$
$$x_2' = -x_1\sin\phi + x_2\cos\phi$$
$$x_3' = x_3$$



1) ROTATION OF $\vec{r}$ BY $-\phi$
"ACTIVE"

2) ROTATION OF AXES BY $+\phi$
"PASSIVE"



MATRIX PROPERTIES

TEXTBOOK NOTATION IS POOR

DOES NOT DISTINGUISH $\vec{x}$ AND $\vec{x}^T$

$\vec{x}$ AND $\overleftrightarrow{A}$

$$\vec{x}^T \equiv \tilde{\vec{x}}$$

SHOWN $\overleftrightarrow{A}\,\overleftrightarrow{A}^{-1} = \overleftrightarrow{A}\,\overleftrightarrow{A}^T = \overleftrightarrow{\mathbb{1}}$

ONLY FOR ROTATIONS (AND SOME OTHERS)

$$|\overleftrightarrow{A}\,\overleftrightarrow{B}| = |\overleftrightarrow{A}|\,|\overleftrightarrow{B}|$$

$$1 = |\overleftrightarrow{\mathbb{1}}| = |\overleftrightarrow{A}|\,|\overleftrightarrow{A}^T| = |\overleftrightarrow{A}|^2 \qquad \cancel{\text{de}}\ |\overleftrightarrow{A}| = \pm 1$$

if $|\overleftrightarrow{A}| = -1$ ~~NON PHYSICAL~~
NOT A ROTATION

$$\overleftrightarrow{S} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -\overleftrightarrow{\mathbb{1}} \qquad |\overleftrightarrow{S}| = -1$$

ROTATE BY π

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

REFLECT

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

a) REFLECTION TURNS A RIGHT HANDED COORD SYSTEM INTO A LH SYSTEM

b) CANNOT BE DONE INFINITESIMALLY

DESCRIBE SYSTEM ~~EULER~~ w/ 3 DOF

a) ~~AN~~ EULER ANGLES

1) ROTATE AROUND z BY ϕ $(\overleftrightarrow{D})$

2) " " NEW x BY θ $(\overleftrightarrow{C})$

3) " " NEWER z BY ψ $(\overleftrightarrow{B})$

$$D = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$B = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \overleftrightarrow{A} = \overleftrightarrow{B}\,\overleftrightarrow{C}\,\overleftrightarrow{D}$$

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b) ROLL PITCH YAW

c) $\hat{n}, \phi$ THE GENERAL DISPLACEMENT OF A RIGID BODY WITH ONE FIXED POINT IS A ROTATION AROUND SOME AXIS

$$\forall\, \overleftrightarrow{R}\ \exists\ \hat{n} \ni \overleftrightarrow{R}\hat{n} = \hat{n}$$

$$\vec{r}\,' = \overleftrightarrow{R}\vec{r} = \lambda\vec{r} \qquad \overleftrightarrow{\lambda} = \lambda\,\overleftrightarrow{\mathbb{1}}$$

$$(\overleftrightarrow{R} - \overleftrightarrow{\lambda})\vec{r} = 0 \Rightarrow |\overleftrightarrow{R} - \overleftrightarrow{\lambda}| = 0$$

EIGENVALUES λ_i 3 ROOTS (IN 3D)

EIGENVECTORS $\vec{a}_i$

$\overleftrightarrow{A} = (\vec{a}_1, \vec{a}_2, \vec{a}_3)$ ASSEMBLING 3 COLUMN EIGENVECTORS TO A MATRIX

$\overleftrightarrow{R}\,\vec{r} = \lambda \vec{r}$

$\overleftrightarrow{\lambda} = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix}$ (CHANGED)

$\overleftrightarrow{R}\,\overleftrightarrow{A} = \overleftrightarrow{A}\,\overleftrightarrow{\lambda}$

$\overleftrightarrow{A}^{-1}\overleftrightarrow{R}\,\overleftrightarrow{A} = \overleftrightarrow{\lambda} \Rightarrow |\overleftrightarrow{A}^{-1}\overleftrightarrow{R}\,\overleftrightarrow{A}| = |\overleftrightarrow{\lambda}| = \lambda_1\lambda_2\lambda_3$

$1 = |\overleftrightarrow{A}^{-1}|\,|\overleftrightarrow{R}|\,|\overleftrightarrow{A}| = \text{''}$

IS THERE A SOLUTION?

CONSIDER $(\overleftrightarrow{A} - \overleftrightarrow{\mathbb{1}})(\overleftrightarrow{A}^T) = (\overleftrightarrow{\mathbb{1}} - \overleftrightarrow{A}^T)$

$\underline{|\overleftrightarrow{A} - \overleftrightarrow{\mathbb{1}}|}\,\underbrace{|\cancel{\overleftrightarrow{A}^T}|}_{1} = \underline{|\overleftrightarrow{\mathbb{1}} - \overleftrightarrow{A}|}$

BUT $|-\overleftrightarrow{B}| = (-1)^n |\overleftrightarrow{B}|$

IF $n = 3$ $\quad |\overleftrightarrow{A} - \overleftrightarrow{\mathbb{1}}| = |\overleftrightarrow{\mathbb{1}} - \overleftrightarrow{A}|$ IFF

$|\overleftrightarrow{A} - \overleftrightarrow{\mathbb{1}}| = 0$

$|\overleftrightarrow{A} - \lambda\overleftrightarrow{\mathbb{1}}| = 0$ IF $\lambda = 1$

THUS $\lambda = 1$ MUST BE AN EIGENVALUE (ONLY IN 3D)

CHOOSE $\lambda_3 = 1$ $\quad \lambda_1\lambda_2 = 1$

$\lambda_1 = e^{i\phi} \quad \lambda_2 = e^{-i\phi}$

a) $\lambda_1 = \lambda_2 = \lambda_3 = 1$ THEN $\overleftrightarrow{R} = \overleftrightarrow{\mathbb{1}}$ TRIVIAL

b) $\lambda_3 = 1$, $\lambda_1 = \lambda_2 = -1$ $(\phi = \pi)$

c) $\lambda_3 = 1$ $\lambda_1 = \lambda_2^* = e^{i\phi}$

FIND $\hat{n}$ BY CHOOSING $\lambda = 1$ AND SOLVING

$$(\overleftrightarrow{A} - \lambda \overleftrightarrow{\mathbb{1}})\hat{n} = 0$$

THEN USE A SIMILARITY TRANSFORM $(\overleftrightarrow{B}^{-1}\overleftrightarrow{A}\overleftrightarrow{B})$
SO NEW $z' \parallel \hat{n}$

$$(\overleftrightarrow{A}') = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{Tr}\,\overleftrightarrow{A}' = 1 + 2\cos\phi$$

$$\mathrm{Tr}\,\overleftrightarrow{A}' = \mathrm{Tr}\,\overleftrightarrow{A} \qquad \cos\phi = \frac{1}{2}\left(\mathrm{Tr}\,\overleftrightarrow{A} - 1\right)$$

$$\mathrm{Tr}\,A = \sum \lambda_i = 1 + e^{-i\phi} + e^{i\phi}$$

1) AMBIGUITY $\hat{n}$ or $-\hat{n}$

ϕ or $-\phi$

CHASLES' THM: THE MOST GENERAL DISPLACEMENT OF A RIGID BODY IS A TRANSLATION PLUS A ROTATION