INFINITESIMAL ROTATIONS

$$\overleftrightarrow{A}\,\overleftrightarrow{B} \neq \overleftrightarrow{B}\,\overleftrightarrow{A} \qquad \text{FINITE}$$

$$x_1' = x_1 + \epsilon_{11}x_1 + \epsilon_{12}x_2 + \epsilon_{13}x_3$$

$$x_i' = x_i + \epsilon_{ij}x_j$$

$$\vec{x}\,' = \left(\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}\right)\vec{x}$$

SMALL

$$\left(\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}_1\right)\left(\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}_2\right) = \overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}_1\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\mathbb{1}}\,\overleftrightarrow{\epsilon}_2 + \cancel{\overleftrightarrow{\epsilon}_1\overleftrightarrow{\epsilon}_2}$$

$$= \overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}_1 + \overleftrightarrow{\epsilon}_2$$ COMMUTES!

EULER ROTATIONS z, THEN x', THEN z'

$$z: \begin{pmatrix} 1 & d\phi & 0 \\ -d\phi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad x': \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d\theta \\ 0 & -d\theta & 1 \end{pmatrix} \qquad z': d\psi$$

EULER MATRIX
$$\begin{pmatrix} 1 & d\phi + d\psi & 0 \\ -(d\phi + d\psi) & 1 & d\theta \\ 0 & -d\theta & 1 \end{pmatrix} = \overleftrightarrow{A}$$

$$d\vec{\Omega} = \hat{i}\,d\theta + \hat{k}\,(d\theta + d\psi)$$

$$\overleftrightarrow{A} = \overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon} \qquad \overleftrightarrow{A}^{-1} = \overleftrightarrow{\mathbb{1}} - \overleftrightarrow{\epsilon}$$

$$\overleftrightarrow{A}\,\overleftrightarrow{A}^{-1} = \left(\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}\right)\left(\overleftrightarrow{\mathbb{1}} - \overleftrightarrow{\epsilon}\right) = \overleftrightarrow{\mathbb{1}}$$

$$\overleftrightarrow{A}^{-1} = \overleftrightarrow{A}^{T} \Rightarrow -\overleftrightarrow{\epsilon} = \overleftrightarrow{\epsilon}^{T}$$ ANTISYMMETRIC

$\Rightarrow$ 3 DOF

$$\overleftrightarrow{\epsilon} = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\vec{r}\,' - \vec{r} = d\vec{r} = \overleftrightarrow{\epsilon}\,\vec{r}$$

$$dx_1 = x_2\, d\Omega_3 - x_3\, d\Omega_2$$

$$dx_2 = -x_1\, d\Omega_3 + x_3\, d\Omega_1$$

$$dx_3 = x_1\, d\Omega_2 - x_2\, d\Omega_1$$

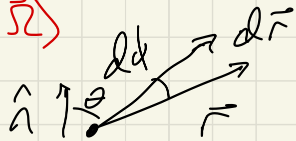
CROSS PRODUCTS!

$$d\vec{r} = \vec{r} \times \underline{d\vec{\Omega}}$$

THIS IS VECTOR OF INFINITESIMAL LENGTH NOT A $d(\vec{\Omega})$

IS THIS A VECTOR?

ROTATE $\vec{r}$ by $d\phi$ AROUND $\hat{n}$



$$|d\vec{r}| = r \sin\theta\, d\phi$$

$$d\vec{r} \perp \vec{r} \qquad d\vec{r} \perp \hat{n}$$

$$\therefore\ d\vec{r} = \vec{r} \times \hat{n}\, d\phi = \vec{r} \times d\vec{\Omega}$$

$\hat{n}$ IS A VECTOR $\Rightarrow$ $d\vec{\Omega}$ IS A VECTOR

IS $d\vec{\Omega}$ AXIAL OR POLAR?

$X \Rightarrow -X$

$Y \Rightarrow -Y$

PARITY: $\mathbb{P}$(SCALAR) = SCALAR $\quad Z \to -Z$

$\mathbb{P}$(VECTOR) = $-$VECTOR

$\mathbb{P}$(AXIAL-VECTOR) = $+$(A-V)

AXIAL VECTOR $\vec{L} = \vec{r} \times \vec{p}$

$$\mathbb{P}(\vec{L}) = \mathbb{P}(\vec{r}) \times \mathbb{P}(\vec{p}) = -\vec{r} \times -\vec{p} = +\vec{L}$$

SCALAR $\vec{v} \cdot \vec{v}$

PSEUDO SCALARS $\vec{V} \cdot \vec{V}^*$

POLAR AXIAL

$\vec{r}, d\vec{r}$ POLAR VECTORS

$\vec{r} \times \hat{n}$ IS POLAR VECTOR

$\Rightarrow \hat{n}$ AND $d\vec{\Omega}$ AXIAL VECTORS

$\epsilon_{ijk} = 0$ IF $i=j$ $j=k$ $i=k$

$= 1$ EVEN PERMUTATIONS OF 123

$= -1$ ODD " " "

$$\vec{V}^* = \vec{d} \times \vec{f} \Rightarrow V_i^* = d_j f_k - f_j d_k$$

$$= \epsilon_{ijk} d_j f_k$$

FINITE ROTATIONS

CCW ROTATIONS OF VECTORS

$$\vec{r}' = \vec{r}\cos\phi + \hat{n}(\hat{n}\cdot\vec{r})(1-\cos\phi)$$

$$+ \hat{n} \times \vec{r} \sin\phi$$

$$d\vec{r}' = d\vec{\Omega} \times \vec{r} = (\hat{n} \times \vec{r})\,d\phi \quad -(\vec{r} \times \hat{n})\,d\phi$$

$$\epsilon_1 = \begin{pmatrix} 0 & -d\Omega_3 & d\Omega_2 \\ d\Omega_3 & 0 & -d\Omega_1 \\ -d\Omega_2 & d\Omega_1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}}_{\overleftrightarrow{N}^T} d\phi$$

$$\frac{d\vec{r}}{d\phi} = -\overleftrightarrow{N}\vec{r} \qquad \overleftrightarrow{N} \Rightarrow N_{ij} = \epsilon_{ijk} n_k$$

$$\overleftrightarrow{\epsilon} = n_i \overleftrightarrow{M}_i \, d\phi \quad \text{(implied sum)}$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

INFINITESIMAL ROTATION GENERATORS

$$\text{rot} = \overleftrightarrow{\underline{1}} + \theta M_i$$

$$[\overleftrightarrow{M}_i, \overleftrightarrow{M}_j] = \overleftrightarrow{M}_i \overleftrightarrow{M}_j - \overleftrightarrow{M}_j \overleftrightarrow{M}_i = \epsilon_{ijk} \overleftrightarrow{M}_k$$

RATE OF CHANGE OF VECTOR $\bar{g}$

$$(d\vec{g})_{\text{SPACE}} = (d\vec{g})_{\text{body}} + (d\vec{g})_{\text{rot}}$$

SEEN BY EXTERNAL OBSERVER

SEEN BY OBSERVER ON BODY

ROTATION OF BODY

$$(d\vec{g})_{\text{rot}} = d\vec{\Omega} \times \vec{g}$$

$$\left(\frac{d\vec{g}}{dt}\right)_{\text{space}} = \left(\frac{d\vec{g}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{g}$$

ANGULAR VELOCITY

$$\left(\frac{d}{dt}\right)_{\text{space}} = \left(\frac{d}{dt}\right)_{\text{body}} + \vec{\omega} \times$$

CAN WRITE $\vec{\omega}$ IN TERMS OF $\frac{d}{dt}$ OF THE EULER ANGLES

MOST CONVENIENT IN BODY SYSTEM

ROTATE BY ϕ AROUND z

θ AROUND x'

ψ AROUND z'

TO GO FROM SPACE (UNPRIMED) TO BODY (PRIMED) SYSTEM

ψ AFFECTS ω_z'

θ " ω_x' ω_y'

ϕ " ALL 3

$\omega_z' = \dot{\psi} + \dot{\phi}\cos\theta$ $\quad$ $\omega_{x'} = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$

$\omega_y' = \dot{\theta}(-\sin\psi) + \dot{\phi}\sin\theta\cos\psi$

CORIOLIS EFFECT

B = BODY

r = ROTATING FRAME

$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_{\cancel{b}\,r} + \vec{\omega}\times$$

$$\left(\frac{d\vec{r}}{dt}\right)_s = \left(\frac{d\vec{r}}{dt}\right)_b + \vec{\omega}\times\vec{r}$$

$$\vec{V}_s = \vec{V}_b + \vec{\omega}\times\vec{r}$$

V wrt the stars

V wrt Earth

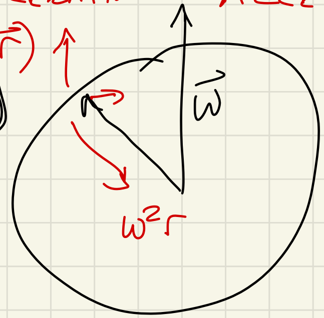
ROTATION OF EARTH

$$\left(\frac{d^2\vec r}{dt^2}\right)_s = \frac{d}{dt}\left(\left(\frac{d\vec r}{dt}\right)_b + \vec\omega\times\vec r\right) + \vec\omega\times\left(\left(\frac{d\vec r}{dt}\right)_b + \vec\omega\times\vec r\right)$$

$$= \left(\frac{d^2\vec r}{dt^2}\right)_b + \omega\times\left(\frac{d\vec r}{dt}\right)_b + \vec\omega\times\left(\frac{d\vec r}{dt}\right)_b + \vec\omega\times\vec\omega\times\vec r$$

$$\vec a_s = \left(\frac{d^2\vec r}{dt^2}\right)_b + \underbrace{2\vec\omega\times\left(\frac{d\vec r}{dt}\right)_b}_{\text{CORIOLIS ACCELERATION}} + \underbrace{\vec\omega(\vec\omega\cdot\vec r) - \omega^2\vec r}_{\text{CENTRIFUGAL ACCEL}}$$

$$\vec a_b = \vec a_s - 2\vec\omega\times\left(\frac{d\vec r}{dt}\right)_b + (\omega^2\vec r - \vec\omega(\vec\omega\cdot\vec r))$$



CENTRIFUGAL
VELOCITY-
INDEPENDENT

CORIOLIS: SIDEWAYS
$\propto V$

HOW BIG IS IT?

$$\omega = \frac{2\pi}{24\ \text{HOURS}} = \frac{2\pi}{86400\,\text{sec}} \simeq 7\times10^{-5}\ \frac{\text{RAD}}{\text{SEC}}$$

CENTRIFUGAL $\vec a$ @ EQUATOR?

$$\vec a = \omega^2\vec r - (\vec\omega\,\cancel{(\vec\omega\cdot\vec r)}) = (7\times10^{-5}\,s^{-1})^2(6\times10^{6}\,m)$$

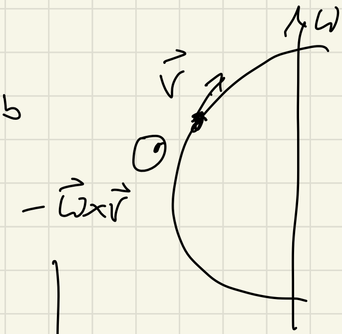
$$= (5\times10^{-9}\,s^{-2})(6\times10^{+6}\,m)$$

$$= 3\times10^{-2}\,m/s^2 \qquad 3\cdot10^{-3}\,g$$

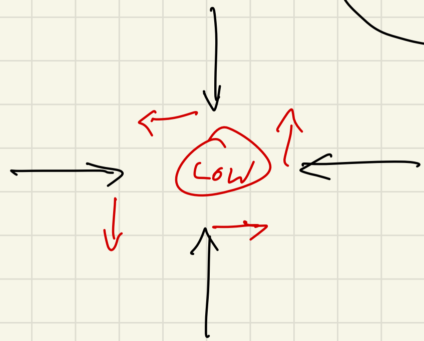
CORIOLIS ACCEL $\quad -2\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_b$

IF $\vec{v}$ IS "NORTH"

THEN $\vec{a}$ IS EAST



HURRICANES:



MISSILE $\quad V = 10^3\ m/s$

$1\ m/s = 2.2\ mi/hr$

$\vec{a}_c = -2\vec{\omega}\times\vec{v}_s = -2(7\times10^{-5}\ s^{-1})\ 10^3\ m/s\ 0.7$

$a_c \simeq 10^{-2}\ m/s^2$

2 MINUTE $\simeq 10^2\ s$ $\qquad \Delta x = \frac{1}{2}at^2$

$= \frac{1}{2}(10^{-2}\ m/s^2)\ 10^4\ s^2$

$\simeq 10^2\ m$

20 MINUTE $\quad 10^3\ s \Rightarrow 10^4\ m$

PULSAR $\quad r = 10\ km \quad T = 10\ ms$

$\omega = 2\pi\frac{1}{T} = 6\times10^2\ s^{-1} \quad (10^7\times\omega_{EARTH})$

CENTRIFUGAL ~~FORCE~~ ACCEL $\quad \omega^2 r = (6\times10^{2}\ s^{-1})(10^4\ m)$

$= 4\times10^9\ m/s^2$

GRAV FORCE : $10^{12}\ m/s^2$

CORIOLIS FORCE 10^7 TIMES LARGER