Solving Rigid Body Problems

non-holonomic constraints $\Rightarrow$ then use Lagrange multipliers (rolling)

holonomic constraints $f(\{x\ q_i\}) = 0$
then choose independent $\{q_i\}$

if axis of rotation is fixed, angles are easy
if not fixed

a) fixed point - use that for origin
b) no fixed point - use cm for origin

use principal axes for coordinate system (body system)

$$\left.\frac{d\vec{L}}{dt}\right)_s = \left.\frac{d\vec{L}}{dt}\right)_{body} + \vec{w} \times \vec{L} = \vec{N}$$

$L_j = I_j w_j$ (no summation)

~~TB~~. work in body system (mostly)

$$\frac{dL_i}{dt} + \epsilon_{ijk} w_j L_k = N_i$$

$$\frac{dL_1}{dt} + w_2 L_3 - w_3 L_2 = N_1$$

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1$$

$$I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = N_2$$

$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3$$

Cannot spin w/o Torque at constant $\vec{\omega}$ unless $\vec{\omega} \parallel \hat{n}_i$ (a principal axis)

Tire balancing: need principal axis $\parallel$ to axis of rotation

Specific case, Torque-free motion $\vec{N} = 0$
symmetric object $I_1 = I_2$

$$I_3 \dot{\omega}_3 = 0 \qquad \begin{aligned} \dot{\omega}_1 + \beta \omega_3 \omega_2 &= 0 \\ \dot{\omega}_2 - \beta \omega_3 \omega_1 &= 0 \end{aligned} \qquad \beta = \frac{I_3 - I_1}{I_1}$$

$$\ddot{\omega}_1 + \beta \omega_3 \dot{\omega}_2 = \ddot{\omega}_1 + \beta^2 \omega_3^2 \omega_1 = 0$$

$$\Rightarrow \omega_1 = A \cos(\beta \omega_3 t + \theta)$$
$$\omega_2 = A \sin(\beta \omega_3 t + \theta)$$

$$\omega = (\omega_3^2 + A^2)^{1/2}$$

Precession! Instantaneous axis of rotation traces a cone in the body system body cone $\frac{1}{2}$ angle α_b $\qquad A = \omega \sin\alpha_b$

$\tan\alpha_b = A/\omega_3 \qquad \omega_3 = \omega\cos\alpha_b$

body is rotating around an axis that is rotating

$\vec{L}$ is constant

$$\cos\alpha_s = \frac{\vec{\omega}\cdot\vec{L}}{\omega L} = \frac{\vec{\omega}^T \overleftrightarrow{I}\vec{\omega}}{\omega L} = \frac{2T}{\omega L} \quad \text{space cone}$$

$\alpha_\beta < \alpha_s$ PROLATE (CIGAR) $I_3 > I_1 = I_2$

$\alpha_\beta > \alpha_s$ OBLATE (DISC) $I_3 < I_1 = I_2$

Nondegenerate $\overleftrightarrow{I}$ $\qquad I_1 \neq I_2 \neq I_3$

rotation is stable if $\vec{\omega} \parallel \hat{n}_i$

perturbations?

a) $\omega_3 \gg \omega_1, \omega_2 \qquad \dot{\omega}_3 \propto \omega_1\omega_2$ 2^{nd} order

$$\omega_1 = A\left[I_2(I_3 - I_2)\right]^{1/2}\cos(\beta\omega_3 t + \theta)$$
$$\omega_2 = A\left[I_1(I_3 - I_1)\right]^{1/2}\sin(\beta\omega_3 t + \theta)$$
$$\beta = \left[\frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}\right]^{1/2}$$

1) $I_3 > I_1, I_2 \Rightarrow \beta \in \mathbb{R}$ stable

2) $I_3 < I_1, I_2 \Rightarrow \beta \in \mathbb{R}$ stable

3) $I_1 < I_3 < I_2 \Rightarrow \beta \in \mathbb{C}$ unstable

tennis racket!

More generally

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3)$$
$$I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1)$$
$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2)$$

$$\vec{\rho} = \hat{n}/\sqrt{I} = \frac{\vec{\omega}}{\omega\sqrt{I}} = \frac{\vec{\omega}}{\sqrt{2T}}$$

$$T = \frac{\vec{\omega} \cdot \vec{L}}{2} = \frac{\vec{\omega}^T \overleftrightarrow{I} \vec{\omega}}{2} = \frac{\omega^2}{2} \hat{n}^T \overleftrightarrow{I} \hat{n} = \frac{1}{2} I \omega^2$$

define $F(\vec{\rho}) = \vec{\rho}^T \overleftrightarrow{I} \vec{\rho} = \rho_i^2 I_i$

$F(\vec{\rho}) = 1$ defines the inertia ellipsoid

as $\hat{n}$ and $\vec{\omega}$ change, $\vec{\rho}$ changes
But, the tip of $\vec{\rho}$ stays on the inertia ellipsoid

$\vec{\nabla} F$ is normal to the ellipsoid

$$\vec{\nabla}_\rho F = 2 \overleftrightarrow{I} \vec{\rho} = \frac{2 \overleftrightarrow{I} \vec{\omega}}{\sqrt{2T}} = \sqrt{\frac{2}{T}}\, \vec{L}$$

$\vec{\omega}$ constrained to move so that the $\perp$ to the ellipsoid points $\parallel$ to $\vec{L}$

$\vec{L}$ is fixed in space
ellipsoid is fixed in the body

ellipsoid moves to keep the connection between $\vec{\omega}$ and $\vec{L}$

distance from the origin of the ellipsoid and the plane tangent to it @ $\vec{\rho}$

$$\frac{\vec{\rho}\cdot\vec{L}}{L} = \frac{\vec{\omega}\cdot\vec{L}}{L\sqrt{2T}} = \frac{\sqrt{2T}}{L} \quad \text{CONSTANT !}$$

$\vec{L} \perp$ invariable plane

ellipsoid rotates (rolls without slipping) on the invariable plane

polhode = curve on the surface of the ellipsoid

herpolhode = curve on the surface of the invariable plane.

$\vec{\rho}$ direction of $\vec{\omega}$

orientation of the ellipsoid $\Leftrightarrow$ orientation of the body

Case: symmetric $I_1 = I_2$

inertia ellipsoid = ellipsoid of rotation

polhode = circle around the symmetry axis

herpolhode = circle

$\vec{\omega}$ precesses around axis of symmetry

Now describe motion of $\vec{L}$ wrt body

$$T = \sum \frac{1}{2}\frac{L_i^2}{I_i} = \text{constant}$$

$$\frac{L_1^2}{2TI_1} + \frac{L_2^2}{2TI_2} + \frac{L_3^2}{2TI_3} = 1 \quad \text{ellipsoid}$$

$$L_1^2 + L_2^2 + L_3^2 = L^2 \quad \text{sphere}$$

$\vec{L}$ must lie on the sphere-ellipsoid intersection

$$\sqrt{2TI_3} \leq L \leq \sqrt{2TI_1} \qquad (I_3 \leq I_2 \leq I_1)$$

Consider stability of motion

L_x $\quad L^2 = 2TI_1 - \epsilon \Rightarrow$ closed figures around L_x

L_z $\quad L^2 = 2TI + \epsilon \Rightarrow$ closed figures around L_z

L_y $\quad$ no closed figures, large excursions! unstable

Symmetric case $I_1 = I_2$

$$\omega_1 = A\cos(\beta\omega_3 t + \theta) \qquad \beta = \frac{I_3 - I_1}{I_1}$$
$$\omega_2 = A\sin(\beta\omega_3 t + \theta)$$

Tidal bulge $\sim 30\,km$ at the equator
$\sim 0.5\%$

$$\frac{I_3 - I_1}{I_1} \simeq 3 \cdot 10^{-3}$$

period $\simeq$ 300 days

expect precession around the axis with period $\simeq$ 300 days

$\Rightarrow$ deviations in "latitude" $\simeq 10m$ $(10^{-6}$

annual variation : seasonal
420-day period. due to precession?