HOLONOMIC CONSTRAINTS

FORCES DERIVED FROM $V(\{\vec{r}_i\})$
OR ARE SPECIFIC VELOCITY-DEPENDENT V

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

n 2^{nd} order Diff Eq

$2n$ INITIAL CONDITIONS

CONFIGURATION SPACE

$\{q_i\}$ n DIMENSIONAL
n INDEPENDENT FUNCTIONS $\{q_i(t)\}$

HAMILTONIAN FORMALISM

$2n$ 1^{st} ORDER EQ OF MOTION
STILL NEED $2n$ INITIAL CONDITIONS
$2n$ EOM WITH $2n$ INDEPENDENT VARIABLES

CANNOT USE CONSTRAINT EQ IN HAMILTONIANS

$$\{q_i\}, \{p_i\} \qquad p_i = \frac{\partial \mathcal{L}(\{q_j\}, \{\dot{q}_j\}, t)}{\partial \dot{q}_i}$$

CANONICAL VARIABLES

$$\mathcal{L}(\{q_j\}\ \{\dot{q}_j\}\ t) \rightarrow H(\{q_j\}\ \{p_j\}\ t)$$

LEGENDRE TRANSFORM $f(x,y)$

$$df = u\,dx + v\,dy \qquad u = \frac{\partial f}{\partial x} \qquad v = \frac{\partial f}{\partial y}$$

NOW TRANSFORM FROM $(x,y) \rightarrow (u,y)$

$$g \equiv f - ux \qquad dg = df - u\,dx - x\,du$$
$$= v\,dy - x\,du$$

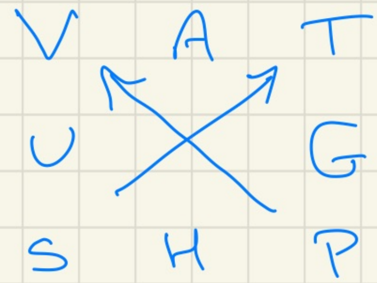
DONE

THERMODYNAMICS

$$U = U(S,V) \qquad U = dQ - dW = T\,dS - p\,dV \qquad T = \frac{\partial U}{\partial S} \qquad P = -\frac{\partial U}{\partial V}$$

ENTHALPY $\quad H = U + PV$

$$\Rightarrow \quad H = H(S,p)$$



NOW XFORM FROM $q, \dot{q} \rightarrow q\ p$

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad \dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} \qquad \left(\text{since } \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0\right)$$

$$d\mathcal{L} = \dot{p}_i\, dq_i + p_i\, d\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$H = \dot{q}_i p_i - \mathcal{L}$$

$$dH = \left(\cancel{p_i\, d\dot{q}_i} + \dot{q}_i\, dp_i\right) - \dot{p}_i\, dq_i - \cancel{p_i\, d\dot{q}_i} - \frac{\partial \mathcal{L}}{\partial t} dt$$

$$= \dot{q}_i\, dp_i - \dot{p}_i\, dq_i - \frac{\partial \mathcal{L}}{\partial t} dt$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad -\dot{p}_i = \frac{\partial H}{\partial q_i} \qquad -\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial H}{\partial t}$$

$\mathcal{L}(\{q_i\}\{\dot{q}_i\}\,t) \qquad H(\{q_i\}\{p_i\}\,t) \qquad$ DIFFERENT FUNCTIONS

GENERAL CONSTRUCTION

1) CONSTRUCT $\mathcal{L} = T - V$
2) $p_i = p_i(\{q_j\}\{\dot{q}_j\}t) = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$
3) $H = \dot{q}_i p_i - \mathcal{L}$ MIXED FN $H(\{q_j\}\{\dot{q}_j\}\{p_j\}t)$
4) SOLVE FOR $\dot{q}_i(\{q_j\}\{p_j\}t)$ CAN BE HARD
5) WRITE TO GET $H(\{q_j\}\{p_j\}t)$

IN PRACTICE IF $\mathcal{L}$ 2ND ORDER IN $\{\dot{q}_j\}$

$$H = \dot{q}_i p_i - \mathcal{L} = \dot{q}_i p_i - \left[L_0(\{q_j\}t) + L_1^k(\{q_j\}t)\dot{q}_k + L_2^{km}(\{q_j\}t)\dot{q}_k\dot{q}_m\right]$$

IF TIME INDEPENDENT: $L_2^{km}\dot{q}_k\dot{q}_m = T$

IF $\vec{F} = -\vec{\nabla} V$ THEN $L_0 = -V$

$$H = T + V = E$$

WE SKIP MOST OF STEPS 3, 4

SIMPLIFY FURTHER: NO OFFDIAGONAL TERMS

$$\mathcal{L} = \mathcal{L}_0 + \dot{q}_i a_i(\{q_j\}t) + \dot{q}_i^2 T_i(\{q_j\}t)$$

<OR> $\mathcal{L} = \mathcal{L}_0 + \dot{\vec{q}}^T\vec{a} + \frac{1}{2}\dot{\vec{q}}^T \overleftrightarrow{\Pi}\dot{\vec{q}}$

EG: $\frac{1}{2}\dot{\vec{q}}^T\overleftrightarrow{\Pi}\dot{\vec{q}} = \frac{1}{2}(\dot{x}\ \dot{y}\ \dot{z})\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \frac{1}{2}m\vec{v}^2$

$$H = \dot{q}^T(\vec{p}-\vec{a}) - \frac{1}{2}\dot{q}^T \overleftrightarrow{\mathbb{T}}\, \dot{q} - \mathcal{L}.$$

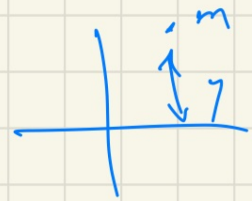
$\overleftrightarrow{\mathbb{T}}$ SYMMETRIC

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Rightarrow \vec{p} = \overleftrightarrow{\mathbb{T}}\,\dot{q} + \vec{a} \qquad \left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

STEP 4 $\dot{q} = \overleftrightarrow{\mathbb{T}}^{-1}(\vec{p}-\vec{a})$

STEP 5 $H(\{q_i\}\{p_i\}t) = \frac{1}{2}(\vec{p}^T - \vec{a}^T)\overleftrightarrow{\mathbb{T}}^{-1}(\vec{p}-\vec{a}) - \mathcal{L}.$

EXAMPLE $\quad T = \frac{1}{2}m\dot{y}^2 \;\&\; V = mgy$



$$\mathcal{L} = \frac{1}{2}m\dot{y}^2 - mgy$$

ELEM : $p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y} \qquad \dot{p}_y = m\ddot{y} = \frac{\partial \mathcal{L}}{\partial y} = -mg$

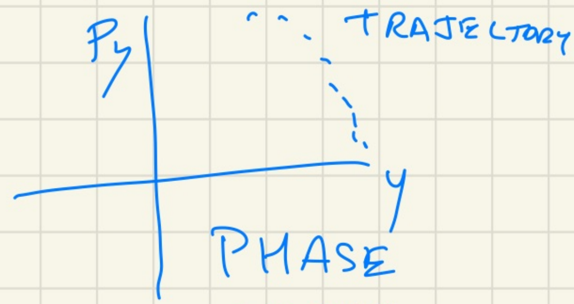
$h = p_y\dot{y} - \mathcal{L} = m\dot{y}^2 - \frac{1}{2}m\dot{y}^2 + mgy = \frac{1}{2}m\dot{y}^2 + mgy$

$h = E$ CONSERVED $\qquad h \neq H$

$H(q,p,t) = \dot{q}_y p_y - \mathcal{L} \qquad \dot{y} = p_y/m$

$$H = p_y^2/2m + mgy$$

HEM : $\frac{\partial H}{\partial p_y} = \frac{p_y}{m} = \dot{y} \qquad -\frac{\partial H}{\partial y} = -mg = \dot{p}_y$



PHASE SPACE

EXAMPLE $V(r)$ SPHERICAL COORDS

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

$\mathbb{T}$ IS DIAGONAL $\begin{bmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2\sin^2\theta \end{bmatrix}$

$$H(r,\theta,\phi,p_r,p_\theta,p_\phi) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}$$

$$H(\vec{x},\vec{p}) = \frac{\vec{p}^{\,2}}{2m} + V(|\vec{x}|)$$

$p_\theta \neq \theta$ COMPONENT OF $\vec{p}$

p_r, p_θ, p_ϕ DO NOT FORM A VECTOR

THEY TRANSFORM DIFFERENTLY THAN (p_x, p_y, p_z)

p_r, p_θ, p_ϕ HAVE DIFFERENT UNITS

p_r HAS UNITS OF LINEAR MOM

p_θ, p_ϕ " " " ANGULAR "

EXAMPLE: NONRELATIVISTIC PARTICLE IN AN EM FIELD

$$\mathcal{L} = T - V = \underbrace{\tfrac{1}{2}mv^2 - q\phi}_{\mathcal{L}_0} + \underbrace{q\vec{A}\cdot\vec{V}}_{\mathcal{L}_1}$$

$$= \frac{m}{2}\dot{x}_i\dot{x}_i + qA_i\dot{x}_i - q\phi$$

$$\vec{P} = \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A}(\vec{r},t)$$

$$P_i = \frac{\partial\mathcal{L}}{\partial\dot{x}_i} = m\dot{x}_i + qA_i(\vec{r},t)$$

STEP 3 $\quad H = P_i\dot{x}_i - \mathcal{L}$

$$= m\dot{x}_i\dot{x}_i + \cancel{qA_i\dot{x}_i} - \left(\frac{m}{2}\dot{x}_i\dot{x}_i + \cancel{qA_i\dot{x}_i} - q\phi\right)$$

$$= \tfrac{1}{2}m\dot{x}_i\dot{x}_i + q\phi \quad \text{(WRONG VARIABLES)}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial\mathcal{L}}{\partial t} \neq 0$$ IF E, B ~~ct~~ TIME DEPENDENT

(PARTICLE EXCHANGES ENERGY W/FIELDS)

$$\dot{x}_i = \left(\frac{P_i}{m} - \frac{qA_i}{m}\right)$$

STEP 4

STEP 5 $\quad H(\vec{p},\vec{r},t) = \frac{1}{2m}\left(\vec{p} - q\vec{A}(\vec{r},t)\right)^2 + q\phi(\vec{r},t)$

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi$$

HEM: $\underbrace{\frac{\partial H}{\partial p_i} = \dot{x}_i}$ $\quad$ $\underbrace{\frac{\partial H}{\partial x_i} = -\dot{p}_i}$

$$(p_i - qA_i) = m\dot{x}_i$$

$$\dot{p}_i = -q\frac{\partial \phi}{\partial x_i} + \frac{q}{m}\left(p_j - qA_j\right)\frac{\partial}{\partial x_i}A_j$$

$$= -q\frac{\partial \phi}{\partial x_i} + q\dot{x}_j\frac{\partial A_j}{\partial x_i}$$

$$m\ddot{x}_i = \dot{p}_i - q\frac{d}{dt}A_i$$

$$= -q\frac{\partial \phi}{\partial x_i} + q\dot{x}_j\frac{\partial A_j}{\partial x_i} - q\frac{dA_i}{dt}$$

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial x_j}\dot{x}_j + \frac{\partial A_i}{\partial t}$$

$$m\ddot{x}_i = -q\frac{\partial \phi}{\partial x_i} + q\frac{\partial A_j}{\partial x_i}\dot{x}_j - q\frac{\partial A_i}{\partial x_j}\dot{x}_j - q\frac{\partial A_i}{\partial t}$$

$$= -q\underbrace{\left(\frac{\partial \phi}{\partial x_i} + \frac{\partial A_i}{\partial t}\right)}_{E_i} + q\,\underbrace{\dot{x}_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)}_{\left[\dot{\vec{r}} \times (\vec{\nabla} \times \vec{A})\right]_i}$$

$$m\ddot{x}_i = q\left[E(\vec{r},t) + \vec{v} \times \vec{B}(\vec{r},t)\right]$$

PHEW!