HAMILTONIANS + CYCLIC COORDS

$$\dot{p}_j = \frac{\partial \mathcal{L}}{\partial q_j} = -\frac{\partial H}{\partial q_j}$$

IF $\mathcal{L}$ IS CYCLIC IN q_j, SO IS H

TRIVIAL SINCE $\mathcal{H} = p_i \dot{q}_i - \mathcal{L}$

TRANSLATION INVARIANCE $\Rightarrow$ $\vec{P}_{cm}$ IS CONSERVED

ROTATION INVARIANCE $\Rightarrow$ L CONSERVED

IF $\mathcal{L} \neq \mathcal{L}(t) \Rightarrow H$ CONSERVED

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}$$

$$= \cancel{-\dot{p}_i \dot{q}_i} + \cancel{\dot{q}_i \dot{p}_i} + \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} = 0$$

WE SHOWED BEFORE THAT IF THE EQ OF TRANSFORMATION THAT DEFINES THE GENERALIZED COORDS IS TIME INDEPENDENT (NO ROTATING COORDS OR TIME DEPENDENT CONSTRAINTS)

AND $V = V(\{q_i\}) \neq V(\{\dot{q}_i\})$

THEN $H = T + V$

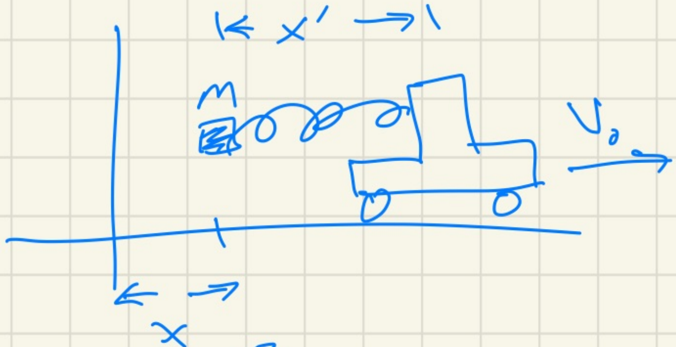
a) H CONSERVED AND/OR b) $H = T + V$

$\mathcal{L} = T - V$ CHANGE VARIABLES $\Rightarrow$ CHANGE THE FORM
MAGNITUDE WON'T CHANGE

$H = p_i \dot{q}_i - \mathcal{L}$ CHANGE VARIABLES $\Rightarrow$ CHANGE p_i

$\Rightarrow$ CHANGE H

EXAMPLE



x:

$$\mathcal{L}(x, \dot{x}, t) = T - V$$

$$= \frac{m\dot{x}^2}{2} - \frac{k}{2}(x - V_0 t)^2$$

~~(mass starts at $x=0$, $t=0$)~~
(spring unstretched for $x=0$, $t=0$)

$$m\ddot{x} = -k(x - V_0 t)$$

CHANGE VARIABLES $x' = x - V_0 t$

$$m\ddot{x}' = -kx'$$

$H(x, p, t)$ $V \neq V(\dot{x})$

$$H(x, p, t) = T + V = \frac{p^2}{2m} + \frac{k}{2}(x - V_0 t)^2$$

$H = T + V$, BUT IT IS NOT CONSERVED

CONSTRAINT IS DOING WORK TO KEEP THE CART MOVING AT CONSTANT V_0 DESPITE ~~OS~~ OSCILLATING MASS

STAY IN REST FRAME, BUT USE x'

$$\mathcal{L}(x', \dot{x}') = \frac{1}{2}m(\dot{x}' + v_0)^2 - \frac{k}{2}x'^2$$

$$= \frac{1}{2}m\dot{x}'^2 + m\dot{x}'v_0 + \frac{mv_0^2}{2} - \frac{k}{2}x'^2$$

HAS TERM LINEAR IN $\dot{x}'$

$$H(\{q\}\{p\}t) = \frac{1}{2}(\vec{p}^T - \vec{a}^T)\overleftrightarrow{T}^{-1}(\vec{p} - \vec{a}) - \mathcal{L}_0$$

$\vec{a}$ = linear terms $\overleftrightarrow{T}$ mass matrix

$\mathcal{L}_0$ DOES NOT DEPEND ON $\{\dot{x}\}$

$$H(x, p, t) = \frac{1}{2}(p-a)\frac{1}{m}(p-a) - \mathcal{L}_0$$

$$H'(x', p') = \frac{1}{2m}(p' - mv_0)^2 - \left(\frac{mv_0^2}{2} - \frac{k}{2}x'^2\right)$$

$H' \neq T + V = E$ DUE TO LINEAR $\dot{x}'$ TERM

BUT H' IS NOW CONSERVED SINCE $\mathcal{L} \neq \mathcal{L}(t)$

H AND H' HAVE DIFFERENT VALUES, DIFFERENT t DEPENDENCE AND DIFFERENT FUNCTIONAL FORM

$\mathcal{L}(x, \dot{x})$ AND $\mathcal{L}(x', \dot{x}')$ HAVE SAME MAGNITUDE + DEPENDENCE DIFFERENT FORM

ALL GIVE SAME PARTICLE MOTION

COMPARE TO  OSCILLATING DUMBBELL

m_1 m_2

MAKING m_2 MOVE WITH CONSTANT V_0 REQUIRES AN PERIODIC EXTERNAL FORCE

CRAZY EXAMPLE

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

$$x = A\cos\omega t$$

$$p = m\dot{x} = -m\omega A \sin\omega t$$

USE $\phi = \omega t = \text{ARCTAN}\left(\frac{-p}{m\omega x}\right)$

$$H = \frac{m\omega^2 A^2 \sin^2\phi}{2} + \frac{kA^2\cos^2\phi}{2} = \frac{m\omega^2 A^2}{2}$$

~~EVERYTHING IS~~ CYCLIC IN ϕ

"$\dot{P}_\phi$" $= 0$ "P_ϕ" $= E$

VARIATIONAL PRINCIPLES

HAMILTON'S PRINCIPLE

$$\delta I \equiv \delta \int_{t_1}^{t_2} \mathcal{L} \, dt = 0$$

GOES FROM SPECIFIED $\{q_i(t_1)\}$

TO A SPECIFIED $\{q_i(t_2)\}$

VARYING PATHS IN N-DIM

CONFIGURATION SPACE $\{q_i\}$

NOW:

NEED TO CHANGE TO PHASE SPACE

2N-DIM $\{q_i\}\{p_i\}$

$$\delta I = \delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(\{q_i\}\{p_i\} t) \right) dt = 0$$

$$= \delta \int_{t_1}^{t_2} f(\{q \dot{q} p \dot{p}\} t) \, dt = 0$$

SOLUTION KNOWN
$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) - \frac{\partial f}{\partial q_i} = 0$$

$i = 1 \ldots N$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_i}\right) - \frac{\partial f}{\partial p_i} = 0$$

f DOES NOT DISTINGUISH BETWEEN p AND q

ONLY $\dot{q}_j$ IS THE $p_i \dot{q}_i$ $\quad \frac{\partial f}{\partial \dot{q}_j} = p_j$

ONLY q_j IS IN H

$$\dot{p}_j + \frac{\partial H}{\partial q_j} = 0 \qquad 1^{ST} \text{ HEM}$$

$$\frac{\partial f}{\partial \dot{p}_j} = 0 \Rightarrow \frac{\partial f}{\partial p_j} = 0$$

$$\dot{q}_j - \frac{\partial H}{\partial p_j} = 0 \qquad 2^{nd} \text{ HEM}$$

THIS VARIATION KEEPS ENDPOINTS FIXED IN SPACE AND TIME

PRINCIPLE OF LEAST ACTION

BEFORE: $\delta q_i(t_1) = \delta q_i(t_2) = 0 \quad \forall i$

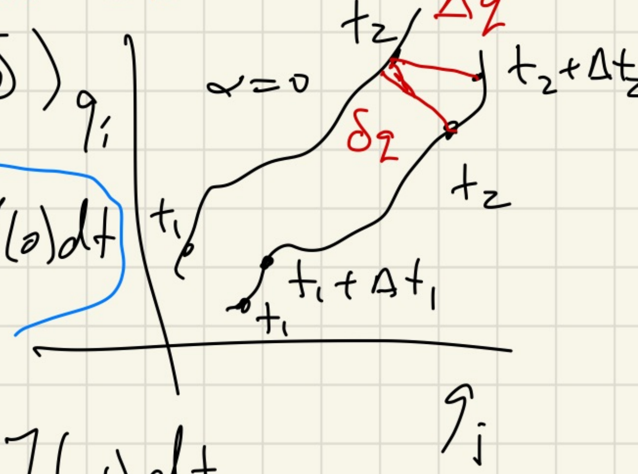
REMOVE THIS CONSTRAINT

$$q_i(t,\alpha) = q_i(t,0) + \alpha\, \eta_i(t)$$

BEFORE: $\eta_i(t_1) = \eta_i(t_2) = 0$

NOW NOT CONSTRAINED

Δ VARIATION (NOT δ)



$$\Delta \int_{t_1}^{t_2} \mathcal{L}\,dt = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \mathcal{L}(\alpha)\,dt - \int_{t_1}^{t_2} \mathcal{L}(0)\,dt$$

$$\Delta \int_{t_1}^{t_2} \mathcal{L}\,dt = \left(\int_{t_2}^{t_2+\Delta t_2} - \int_{t_1}^{t_1+\Delta t_1}\right) \mathcal{L}(\alpha)\,dt$$

$$+ \int_{t_1}^{t_2} \left(\mathcal{L}(\alpha) - \mathcal{L}(0)\right) dt$$

SAME AS BEFORE BUT ENDPOINTS CAN VARY

$$= \mathcal{L}(t_2)\Delta t_2 - \mathcal{L}(t_1)\Delta t_1 + \int_{t_1}^{t_2} \delta \mathcal{L}\,dt$$

$$\int_{t_1}^{t_2} \delta \mathcal{L}\, dt = \underbrace{\int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right] \delta q_i \, dt}_{\text{VANISHES}} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2}$$

new term

$$\Delta \int_{t_1}^{t_2} \mathcal{L}\, dt = \left(\mathcal{L} \Delta t + p_i \delta q_i \right) \Big|_{t_1}^{t_2}$$

δq_i = CHANGE IN $q_i(t)$ @ t_1 AND t_2 SAME TIMES

$$\Delta q_i = \delta q_i + \dot{q}_i \Delta t$$

NOW REQUIRE $\Delta q_i = 0$ SO WE HAVE SAME START AND POINTS

$$\Delta \int_{t_1}^{t_2} \mathcal{L}\, dt = \left(\mathcal{L} \Delta t - p_i \dot{q}_i \Delta t + p_i \Delta q_i \right) \Big|_1^2$$

$$= \left(\cancel{p_i \Delta q_i} - H \Delta t \right) \Big|_1^2$$

$\Delta q_i = 0$

REQUIRE $\mathcal{L} \neq \mathcal{L}(t) \Rightarrow H \neq H(t)$

$\Rightarrow H$ CONSERVED

REQUIRE H CONSERVED ON ALL PATHS

SAME START + END POINTS, BUT POSSIBLY DIFFERENT TIMES

$$\Rightarrow \Delta \int_{t_1}^{t_2} \mathcal{L}\, dt = -H \left(\Delta t_2 - \Delta t_1 \right)$$

$$\int_{t_1}^{t_2} \mathcal{L}\, dt = \int_{t_1}^{t_2} p_i \dot{q}_i \, dt - H (t_2 - t_1)$$

$$\left[\mathcal{L} = p_i \dot{q}_i - H \right]$$

$$\Delta \int_{t_1}^{t_2} \mathcal{L}\, dt = \Delta \int_{t_1}^{t_2} p_i \dot{q}_i\, dt - H(\Delta t_2 - \Delta t_1)$$

(FROM $\mathcal{L} = p_i \dot{q}_i - H$)

H CONSERVED

$$\boxed{\Delta \int_{t_1}^{t_2} p_i \dot{q}_i\, dt = 0}$$

PRINCIPLE OF LEAST ACTION

THIS IS IN CONFIGURATION SPACE $\{q_i\}$

WHAT DOES THIS MEAN

NONREL, TIME INDEP COORDS

$$T = \frac{1}{2} M_{jk}(q)\, \dot{q}_j \dot{q}_k$$

IF $V \neq V(\{\dot{q}_i\})$

THEN $\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} = \dfrac{\partial T}{\partial \dot{q}_i} = p_i$

$$\Rightarrow \quad p_i \dot{q}_i = 2T$$

PRINCIPLE OF LEAST ACTION

$$\Delta \int_{t_1}^{t_2} T\, dt = 0$$

CASE $V = 0$ THEN $T =$ CONSTANT

$\Rightarrow$ MINIMIZE $t_2 - t_1$

IF NO EXTERNAL FORCES, OBJECTS

TRAVEL IN GEODESICS IN CONFIGURATION SPACE