Hamiltonians + Cyclic Coords
\[ \dot{p}_j = \frac{\partial L}{\partial \dot{q}_j} = -2H \]

\( L = L \) is cyclic in \( q^j \), so is \( H \)

**TRIVIAL** since \( H = p_i \dot{q}_i - L \)

**TRANSLATION INvariance** \( \Rightarrow \)
\( \dot{P}_c m \) is conserved

**ROTATION INvariance** \( \Rightarrow \) \( L \) conserved

If \( \dot{L} \neq L(+0) \) \( \Rightarrow \) \( H \) conserved
\[ \frac{dH}{dt} = \frac{\partial H}{\partial q^j} \dot{q}_j + \frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial \tau} \]
\[ = -\dot{p}_j \dot{q}_j + \dot{q}_j \dot{p}_i + \frac{\partial H}{\partial \tau} = -\frac{2H}{\dot{\tau}} = 0 \]

We showed before that if the eq. of transformation that defines the generalized coords is time independent (no rotating coords or time dependent constraints)

and \( \mathbf{V} = \mathbf{V}(q^j; \dot{q}_i) \neq \mathbf{V}(q^j; \dot{q}_j) \)

then \( H = T + V \)

(a) \( H \) conserved and/or  
(b) \( H = T + V \)
\[ y = T - V \quad \text{CHANGE VARIABLES} \Rightarrow \]

\[ \text{CHANGE THE FORM} \]

\[ \text{MAGNITUDE WON'T CHANGE} \]

\[ H = \dot{p} \dot{q} - T \quad \text{CHANGE VARIABLES} \]

\[ \Rightarrow \text{CHANGE } p_i \]

\[ \Rightarrow \text{CHANGE } H \]

\[ kx' \rightarrow 1 \]

\[ \begin{array}{c}
\text{EXAMPLE} \\
\text{x:} \\
L(x, x', t) = T - V \\
= \frac{m x'^2}{2} - \frac{k}{2} (x - V_0 t)^2 \\
\text{(mass starts at } x = 0, t = 0) \\
\text{(spring unstretched for } x = 0, t = 0) \\
\text{CHANGE VARIABLES } x' = x - V_0 t \\
m x'' = -k x' \\
\text{CHANGE VARIABLES } x' = x - V_0 t \\
m x'' = -k x' \\
\end{array} \]

\[ H(x, p, t) = V + V(\dot{x}) \]

\[ H(x, p, t) = T + V = \frac{p^2}{2m} + \frac{k}{2} (x - V_0 t)^2 \]

\[ H = T + V, \text{ but it is not conserved} \]

\[ \text{CONSTRAINT IS DOING WORK TO KEEP THE CART MOVING AT CONSTANT } V_0 \]

\[ \text{OSCILLATING MASS DESPITE } \]
STAY IN REST FRAME, BUT USE $x'$

$$L(x', \dot{x}') = \frac{1}{2} m (\dot{x}' + v_0)^2 - \frac{k}{2} x'^2$$

$$= \frac{1}{2} m \dot{x}'^2 + m \dot{x}' v_0 + m v_0^2 \frac{v_0}{2} - \frac{k}{2} x'^2$$

HAS TERM LINEAR IN $\dot{x}'$

$H(x, p, t) = \frac{1}{2} (\dot{p} - \dot{a})^T \sum^{-1} (\dot{p} - \dot{a}) - E_0$

$a = \text{linear terms}$  $\sum$ mass matrix

$E_0$ does not depend on $\dot{x}'$

$H(x, p, t) = \frac{1}{2} (p - a)^T m (p - a) - E_0$

$H'(x', p') = \frac{1}{2} m (p' - m v_0)^2 - \left( \frac{m v_0^2}{2} - \frac{k}{2} x'^2 \right)$

$H' \neq T + V = E$ DUE TO LINEAR $x'$

But $H'$ is now conserved since $L \neq L(+)$

$H$ and $H'$ have different values, different + dependence and different functional form

$Z(x, \dot{x})$ and $Z(x', \dot{x}')$ have same magnitude + dependence different form

All give same particle motion
COMPARE TO OSCILLATING DUMBBELL

MAKING $m_2$ MOVE WITH CONSTANT $V_0$ REQUIRES AN EXTERNAL FORCE

PERIODIC

CRAZY EXAMPLE

\[ H = \frac{p^2}{2m} + \frac{1}{2} kx^2 \]

\[ x = A \cos \omega t \]

\[ p = mx = -mw \omega \sin \omega t \]

USE \( \phi = \omega t = \arctan \left( \frac{-p}{mwx} \right) \)

\[ H = mw^2A^2 \sin^2 \phi + \frac{kA^2 \cos^2 \phi}{2} = \frac{mw^2A^2}{2} \]

EVERYTHING IS CYCLIC IN $\phi$

\[ "\dot{\phi}" = 0 \quad \"\ddot{\phi}" = E \]
VARIATIONAL PRINCIPLES

HAMILTON'S PRINCIPLE

\[ \delta I = \delta \int_{t_1}^{t_2} \mathcal{L} \, dt = 0 \]

Goes from SPECIFIED \( \xi \mathbf{q}; (t_1) \)

TO A SPECIFIED \( \xi \mathbf{q}; (t_2) \)

VARYING PATHS in N-DIM
CONFIGURATION SPACE \( \xi \mathbf{q}; \mathbf{p} \)

NOW:

NEED TO CHANGE TO PHASE SPACE

2N-DIM \( \xi \mathbf{q}; \mathbf{p} \)

\[ \delta I = \delta \int_{t_1}^{t_2} \mathcal{L} \left( \mathbf{p}; \dot{\mathbf{q}}; - \mathbf{H} \right) \, dt = 0 \]

\[ = \delta \int_{t_1}^{t_2} \mathcal{L} \left( \mathbf{q}; \mathbf{p} \right) \, dt = 0 \]

SOLUTION KNOWN

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \]

\[ i = 1 \ldots N \]

\( \mathbf{f} \) DOES NOT DISTINGUISH
BETWEEN \( \mathbf{p} \) AND \( \mathbf{q} \)

ONLY \( \dot{q}_j \) IS THE \( \mathbf{p}; \dot{q}_i \)

\( \frac{\partial \mathcal{L}}{\partial q_j} = \mathbf{p}_j \)

ONLY \( q_j \) IS IN \( \mathbf{H} \)

\[ \dot{p}_j + \frac{\partial H}{\partial q_j} = 0 \]

1ST HEOM
\[ \frac{\Delta}{\Delta \varphi_j} = 0 \Rightarrow \frac{\partial f}{\partial \varphi_j} = 0 \]

\[ \dot{q}_j - \frac{\partial H}{\partial \dot{q}_j} = 0 \]  
2nd HEM

**This variation keeps endpoints fixed in space and time**

**Principle of Least Action**

**Before:** \[ \delta q_i(t_i) = \delta q_i(t_2) = 0 \quad \forall i \]

**Remove this constraint**

\[ q_i(t_1, \alpha) = q_i(t_1, 0) + \alpha \delta q_i(t) \]

**Before:** \[ \dot{z}_i(t_i) = \dot{z}_i(t_2) = 0 \]

**Now not constrained**

**\( \Delta \) variation (not \( \delta \))**

\[ \Delta S_{t_2} \Delta t = \int_{t_i}^{t_i+\Delta t_i} L(z) dt + \int_{t_2}^{t_2+\Delta t_2} L(\delta z) dt \]

\[ \Delta S_{t_i} \Delta t = (S_{t_2} - S_{t_1}) L(z) dt \]

\[ + S_{t_1} (L(z) - L(0)) dt \]

\[ = L(t_2) \Delta t_2 - L(t_1) \Delta t_1 + S_{t_2} \delta L \Delta t \]

**Same as before but endpoints can vary**
\[ S^2_{t_1}, \delta L_{d+} = S^2_{t_1} \left[ \frac{2L}{2\eta_i} - \frac{d}{dt} \left( \frac{2L}{2\eta_i} \right) \right] \delta g_i \eta_i + \frac{2L}{2\eta_i} \delta g_i \eta_i \left|_{t_1} \right. \]

\[ \triangle S^2_{t_1}, \delta L_{d+} = \left( 2 \Delta t + \eta_i \delta g_i \right) \left|_{t_1} \right. \]

\[ \delta g_i = \text{change in } g_i \text{ at } t_1 \text{ and } t_2 \times \text{same times} \]

\[ \triangle g_i = \delta g_i + \delta \eta_i \Delta t \]

**Now require** \( \triangle g_i = 0 \) so we have same start and points.

\[ \triangle S^2_{t_1}, \delta L_{d+} = \left( 2 \Delta t - \eta_i \delta g_i \Delta t + \eta_i \delta \eta_i \right) \left|_{t_1} \right. \]

\[ = \left( \eta_i \delta g_i - H \Delta t \right) \left|_{t_1} \right. \]

\[ \delta g_i = 0 \]

**Require** \( \Delta t \neq \Delta t^{(+)} \Rightarrow \eta_i \neq \eta_i^{(+)} \Rightarrow H \text{ conserved} \)

**Require** \( H \text{ conserved on all paths} \)

**Same start and end points, but possibly different times**

\[ \Rightarrow \triangle S^2_{t_1}, \delta L_{d+} = -H (\Delta t_2 - \Delta t) \]

\[ S^2_{t_1}, \delta L_{d+} = S^2_{t_1}, \eta_i \delta g_i \Delta t - H (t_2 - t_1) \]

\[ \Rightarrow \Delta = \eta_i \delta g_i - H \Delta t \]
\[ \Delta S_{t_{1}+} = \Delta S_{t_{1}+}, p \cdot \dot{q}_{i} \cdot dt - \eta (\Delta t_{2} - \Delta t_{1}) \]

(from \( t = p \dot{q}_{i} - \eta \))

\[ \Delta S_{t_{1}+}, p \cdot \dot{q}_{i} \cdot dt = 0 \]

H conserved

Principle of Least Action

This is in configuration space \( \Sigma q_{i} \)

What does this mean

Non-rel, time independent coords

\[ T = \frac{1}{2} M_{, jk}(q) \dot{q}_{j} \dot{q}_{k} \]

If \( V \neq V(q_{i}, \dot{q}_{i}) \)

Then \[ \frac{\partial V}{\partial q_{i}} = \frac{\partial T}{\partial \dot{q}_{i}} = p_{i} \]

\[ \Rightarrow \quad p_{i} \dot{q}_{i} = 2T \]

Principle of Least Action

\[ \Delta S_{t_{1}+} T dt = 0 \]

Case \( V = 0 \) then \( T = \text{constant} \)

\[ \Rightarrow \quad \text{minimize} \quad t_{2} - t_{1} \]

If no external forces, objects travel in geodesics in configuration space