CANONICAL TRANSFORMATION

H SOLUTIONS ARE TRIVIAL IF ITS CYCLIC IN <u>ALL</u> q_i

$$p_i = \alpha_i \quad (\text{CONSTANTS}) \quad \forall i$$

$$H = H(\{\alpha_i\})$$

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i \quad (\text{CONSTANT})$$

$$\Rightarrow \quad q_i = \omega_i t + \beta_i$$

CHOOSE MOST CONVENIENT COORDS

IF $V = V(r)$ THEN USE (r, θ) NOT (x, y)

BEFORE $Q_i = Q_i(\{q_j\}\, t)$ POINT TRANSFORM IN CONFIGURATION SPACE

NOW

$$\begin{cases} Q_i = Q_i(\{q_j\}\{p_j\}\, t) \\ P_i = P_i((\{q_j\}\{p_j\}\, t) \end{cases}$$

POINT XFORM IN PHASE SPACE

THIS REQUIRES THAT $\exists\ K(\{Q_i\}\{P_i\}\, t)$

$$\ni \quad \frac{\partial K}{\partial P_i} = \dot{Q}_i \qquad \frac{\partial K}{\partial Q_i} = -\dot{P}_i$$

K IS THE TRANSFORMED H

$$\delta \int_{t_1}^{t_2} \left(P_i \dot{Q}_i - K(\{Q_j\}\{P_j\} t) \right) dt = 0$$

$$\delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(\{q_j\}\{p_j\} t) \right) dt = 0$$

SMALL δ VARIATION: END POINTS AND END TIMES ARE FIXED

$$\Rightarrow \lambda \left(p_i \dot{q}_i - H \right) = \left(P_i \dot{Q}_i - K \right) + \frac{dF}{dt}$$

λ is SCALE XFORM

eg: $Q_i = \mu q_i \quad P_i = \nu p_i$

$$\mu\nu \left(p \dot{q}_i - H \right) = P_i \dot{Q}_i - K$$

SCALE XFORMS ARE TRIVIAL

$\Rightarrow$ FOCUS ON "CANONICAL XFORMS" $(\lambda = 1)$

IF $Q_i \; P_i \neq Q_i(t)$ OR $P_i(t)$

"RESTRICTED CANONICAL XFORM"

THE $\frac{dF}{dt}$ DOES NOT CONTRIBUTE TO δI

$$\left(\delta \int_{t_1}^{t_2} \frac{dF}{dt} dt = 0 \right)$$

IF $F = F(q, p, Q, P, t)$

$\delta q_i = \delta p_j = \delta Q_k = \delta P_\ell = 0$ AT ENDPOINTS

F = GENERATING FN

$F = F(q, Q, t)$

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt}$$

$$= P_i \dot{Q}_i - K + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial Q_i} \dot{Q}_i$$

① $\left(P_i + \frac{\partial F}{\partial Q_i}\right) \dot{Q}_i = 0 \quad \Rightarrow P_i(q, Q, t)$

② $\left(p_i - \frac{\partial F}{\partial q_i}\right) \dot{q}_i = 0 \quad \Rightarrow p_i(q, Q, t)$

③ $K = H + \frac{\partial F}{\partial t}$

USE $\{p_i(q, Q, t)\}$, INVERT TO GET $\{Q_i(q, p, t)\}$

THEN USE Q_i TO GET $P_i = -\partial F / \partial Q_i$

$\Rightarrow P_i(q, p, t)$

GET $q_i(Q, P, t)$ AND $p_i(P, Q, t)$

TO GET $K(Q, P, t)$

$F(q, Q\ t)$

$F(q, P, t)$ . . .

GEN FN

$$F = F_1(q, Q, t)$$

$$F = F_2(q, P, t) - Q_i P_i$$

$$F = F_3(p, Q, t) - q_i p_i$$

$$F = F_4(p, P, t) - Q_i P_i - q_i p_i$$

DERIVATIVES

$$p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = \frac{-\partial F_1}{\partial Q_i}$$

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

TRIVIAL CASE

$$F_1 = q_i Q_i$$

$$p_i = Q_i \quad P_i = -q_i$$

$$F_2 = q_i P_i$$

$$p_i = P_i \quad q_i = Q_i$$

RELATED THRU LEGENDRE XFORMS

EXAMPLES OF CT

a) $F_2 = f_i(\{q_j\} t) P_i \qquad F = F_2 - Q_i P_i$

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(\{q_i\} t)$$ POINT XFORM

ALL POINT XFORMS ARE CANONICAL XFORMS

b) $F_2 = f_i(\{q_j\} t) P_i + g(\{q_j\} t)$

$\Rightarrow$ SAME $\{Q_i\}$, BUT DIFFERENT $\{P_i\}$

$$p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial f_i}{\partial q_j} P_i + \frac{\partial g}{\partial q_j}$$

DIFFERENT $\{P_i\}$, DIFFERENT H

P_i STILL CANONICAL TO Q_i

SAME EOM

$$\vec{p} = \frac{\partial \vec{f}}{\partial \vec{q}} \vec{P} + \frac{\partial g}{\partial \vec{q}}$$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} \partial g / \partial q_1 \\ \partial g / \partial q_2 \end{bmatrix}$$

INVERT TO GET P_1, P_2 IN TERMS OF $\{p_i\}$

c) $F_1 = q_i Q_i$ $\qquad p_i = \frac{\partial F_1}{\partial q_i} = Q_i$

$$P_i = \frac{\partial F_1}{\partial Q_i} = -q_i$$

HARMONIC OSCILLATOR

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$$

CT: $p = f(P)\cos Q$

$$q = \frac{f(P)}{m\omega}\sin Q$$

$$\Rightarrow K = H = \frac{f^2(P)}{2m}\left(\cos^2 Q + \sin^2 Q\right)$$

$$= \frac{f^2(P)}{2m}$$

CHOOSE $f(P) \ni$ XFORM IS CANONICAL

$F_1 = \frac{m\omega q^2}{2}\cot Q$ $\qquad p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}$$

$q^2 = \frac{2P\sin^2 Q}{m\omega}$ $\qquad q = \sqrt{\frac{2P}{m\omega}}\sin Q$

$$p = m\omega q \cot Q \qquad q = \sqrt{\frac{2P}{m\omega}} \sin Q$$

$$p = m\omega \sqrt{\frac{2P}{m\omega}} \cos Q \sin Q = \sqrt{2Pm\omega} \cos Q$$

$$f(P) = \sqrt{2Pm\omega}$$

$$H = \frac{f^2(P)}{2m} = \omega P$$

CYCLIC IN Q $\Rightarrow$ $P = \text{CONST}$

$$P = E/\omega \qquad \dot{Q} = \frac{\partial H}{\partial P} = \omega$$

$$Q = \omega t + \alpha$$

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha)$$

$$p = \sqrt{2mE} \cos(\omega t + \alpha)$$