SYMPLECTIC APPROACH

START WITH TIME INDEPENDENT CT

$$Q_i = Q_i(q, p, \cancel{t})$$

$$P_i = P_i(q, p)$$

H DOES NOT CHANGE

$$\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j = \frac{\partial Q_i}{\partial q_j}\underline{\frac{\partial H}{\partial p_j}} - \frac{\partial Q_i}{\partial p_j}\underline{\underline{\frac{\partial H}{\partial q_j}}}$$

$$p_j = p_j(Q, P) \qquad q_j = q_j(Q, P)$$

$$\frac{\partial H}{\partial P_i} = \underline{\frac{\partial H}{\partial p_j}}\frac{\partial p_j}{\partial P_i} + \underline{\underline{\frac{\partial H}{\partial q_j}}}\frac{\partial q_j}{\partial P_i}$$

WANT $\dot{Q}_i = \frac{\partial H}{\partial P_i}$ TRUE IF

$$\left.\frac{\partial Q_i}{\partial q_j}\right)_{qp} = \left.\frac{\partial p_j}{\partial P_i}\right)_{QP} \qquad \left.\frac{\partial Q_i}{\partial p_j}\right)_{qp} = -\left.\frac{\partial q_j}{\partial P_i}\right)_{QP}$$

SYMPLECTIC CONDITION

IF TRUE, THEN XFORM IS CANONICAL

$$\dot{P}_i = \frac{\partial P_i}{\partial q_j}\dot{q}_j + \frac{\partial P_i}{\partial p_j}\dot{p}_j = \frac{\partial P_i}{\partial q_j}\underline{\frac{\partial H}{\partial p_j}} - \frac{\partial P_i}{\partial p_j}\underline{\frac{\partial H}{\partial q_j}}$$

$$\frac{\partial H}{\partial Q_i} = \underline{\frac{\partial H}{\partial p_j}}\frac{\partial p_j}{\partial Q_i} + \underline{\frac{\partial H}{\partial q_j}}\frac{\partial q_j}{\partial Q_i}$$

IF CANONICAL $\dot{P}_i = -\frac{\partial H}{\partial Q_i}$ REQUIRES

$$\left.\frac{\partial P_i}{\partial q_j}\right)_{qp} = -\left.\frac{\partial p_j}{\partial Q_i}\right)_{QP} \qquad \left.\frac{\partial P_i}{\partial p_j}\right)_{qp} = \left.\frac{\partial q_j}{\partial Q_i}\right)_{QP}$$

SHOW THAT $Q = q\cos\alpha - p\sin\alpha$ IS A CT

$P = q\sin\alpha - p\cos\alpha$

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \Rightarrow \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix}$$

$$\frac{\partial Q}{\partial q} = \cos\alpha \stackrel{?}{=} \frac{\partial p}{\partial P} = \cos\alpha \ \checkmark$$

$$\frac{\partial Q}{\partial p} = -\sin\alpha \stackrel{?}{=} -\frac{\partial q}{\partial P} = -\sin\alpha \ \checkmark$$

SWITCH TO COMPACT NOTATION
AND REPEAT

$$\vec{\eta}^{\,T} = (q_1 \cdots q_n\ p_1 \cdots p_n) \qquad \overleftrightarrow{J} = \begin{pmatrix} \overleftrightarrow{0} & \overleftrightarrow{\mathbb{1}} \\ -\overleftrightarrow{\mathbb{1}} & \overleftrightarrow{0} \end{pmatrix}$$

$\overleftrightarrow{0}$, $\overleftrightarrow{\mathbb{1}}$ ARE $n \times n$

$$\dot{\vec{\eta}} = \overleftrightarrow{J} \frac{\partial H}{\partial \vec{\eta}} \qquad \left(\frac{\partial H}{\partial \vec{\eta}}\right)^T = \left(\frac{\partial H}{\partial q_1}, \cdots, \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \cdots\right)$$

$$q_i = \frac{\partial H}{\partial p_i} \qquad p_i = -\frac{\partial H}{\partial q_i}$$

$$\vec{\xi}^{\,T} = (Q_1 \cdots Q_n\ P_1 \cdots P_n)$$

$$\vec{\xi} = \vec{\xi}(\vec{\eta})$$

$$\dot{\xi}_i = \frac{\partial \xi_i}{\partial \eta_j} \dot{\eta}_j \qquad i, j = 1, \cdots, 2n$$

$$\dot{\vec{\xi}} = \overleftrightarrow{M} \dot{\vec{\eta}} \qquad \underline{M_{ij} = \frac{\partial \xi_i}{\partial \eta_j}}$$

$$\dot{\vec{\xi}} = \overleftrightarrow{M} \overleftrightarrow{J} \frac{\partial H}{\partial \vec{\eta}}$$

REVERSE TRANSFORM

$$\frac{\partial H}{\partial \eta_i} = \frac{\partial H}{\partial \xi_j} \frac{\partial \xi_j}{\partial \eta_i}$$

$$\frac{\partial H}{\partial \vec{\eta}} = \overleftrightarrow{M}^T \frac{\partial H}{\partial \vec{\xi}}$$

$$\dot{\vec{\xi}} = \overleftrightarrow{M}\,\overleftrightarrow{J}\,\frac{\partial H}{\partial \vec{\eta}} = \overleftrightarrow{M}\,\overleftrightarrow{J}\,\overleftrightarrow{M}^T \frac{\partial H}{\partial \vec{\xi}}$$

WANT $\dot{\vec{\xi}} = \overleftrightarrow{J}\,\frac{\partial H}{\partial \vec{\xi}}$

TRUE IF $\overleftrightarrow{M}\,\overleftrightarrow{J}\,\overleftrightarrow{M}^T = \overleftrightarrow{J}$ SYMPLECTIC CONDITION

$$\overleftrightarrow{M}\,\overleftrightarrow{J}\,\cancel{\overleftrightarrow{M}^T\,\overleftrightarrow{M}^{T^{-1}}} = \overleftrightarrow{J}\,\overleftrightarrow{M}^{T^{-1}} \qquad \overleftrightarrow{J}^2 = -\mathbb{1}$$

$$\overleftrightarrow{J}\,\overleftrightarrow{M}\,\cancel{\underbrace{\overleftrightarrow{J}(-\overleftrightarrow{J})}} = \underbrace{\cancel{\overleftrightarrow{J}\,\overleftrightarrow{J}}}_{-\mathbb{1}}\,\overleftrightarrow{M}^{T^{-1}}(-\overleftrightarrow{J})$$

$$J M = M^{T^{-1}} J$$

$$M^T J M = J$$

USE $F = F_2(q_1, P_1) + F_1(q_2, Q_2)$

$$= q_1 P_1 + q_2 Q_2$$

$$\vec{\eta} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \qquad \vec{\xi} = \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix}$$

USING TABLE 9.1 $\quad q_1 P_1 \Rightarrow Q_1 = q_1 \quad P_1 = p_1$

$$\dot{\vec{\xi}} = \overleftrightarrow{M}\,\dot{\vec{\eta}} \qquad q_2 Q_2 \Rightarrow Q_2 = p_2 \quad P_2 = -q_2$$

$$\begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 \\ \dot{p}_2 \\ \dot{p}_1 \\ -\dot{q}_2 \end{pmatrix}$$

HAMILTON'S Eq $\dot{\vec{\eta}} = \overleftrightarrow{J}\frac{\partial H}{\partial \vec{\eta}}$ $\dot{\vec{\xi}} = \overleftrightarrow{J}\frac{\partial H}{\partial \vec{\xi}}$

$$\begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\dot{P}_1 \\ -\dot{P}_2 \\ \dot{Q}_1 \\ \dot{Q}_2 \end{pmatrix}$$

CHECK $M^T J M = J$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = J$$

THIS DERIVATION IS TIME INDEPENDENT ONLY. BUT THE CONDITIONS STILL APPLY IF CT IS TIME DEP.

TIME DEP CT $\vec{\xi} = \vec{\xi}(\vec{\eta}, t)$ IS THIS CT?

$$\vec{\xi} = \vec{\xi}(\vec{\eta}, t_0) \quad ✓$$

LOOK AT $\vec{\xi}(t_0) \to \vec{\xi}(t)$ IS THIS CT?

LOOK AT INFINITESIMAL CT (ICT)

$$Q_i = q_i + \delta q_i \quad P_i = p_i + \delta p_i$$

$$\vec{\xi} = \vec{\eta} + \delta\vec{\eta}$$

$$F_2 = \underbrace{q_i P_i}_{\text{IDENTITY GENERATOR}} + \epsilon G(q, P, t)$$

$$P_j = \frac{\partial F_2}{\partial q_j} = P_j + \epsilon \underbrace{\frac{\partial G}{\partial q_j}}$$

$$\delta p_j = -\epsilon \frac{\partial G}{\partial q_j}$$

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \epsilon \frac{\partial G}{\partial P_j} = q_j + \frac{\partial G}{\underline{\partial p_j}}$$

TO 1ST ORDER

$$\delta q_j = \epsilon \frac{\partial G}{\partial p_j}$$

$$\delta \vec{\eta} = \epsilon \overleftrightarrow{J} \frac{\partial G}{\partial \vec{\eta}}$$

IS THIS CT?

WANT ICT $\quad \vec{\xi}(t_0) \to \vec{\xi}(t_0 + dt)$

IF THIS IS CT THEN $\vec{\xi}(t_0) \to \vec{\xi}(t)$ IS CT

$$\overleftrightarrow{M} = \frac{\partial \vec{\xi}}{\partial \vec{\eta}} = \overleftrightarrow{\mathbb{1}} + \frac{\partial \delta \vec{\eta}}{\partial \vec{\eta}} = \overleftrightarrow{\mathbb{1}} + \epsilon \overleftrightarrow{J} \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}}$$

$$\left(\frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} \right)_{ij} = \frac{\partial^2 G}{\partial \eta_i \partial \eta_j}$$

$$A = BC$$

$$A^T = C^T B^T$$

$$\overleftrightarrow{M}^T = \overleftrightarrow{\mathbb{1}} - \epsilon \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} \overleftrightarrow{J}$$

$$\overleftrightarrow{J}^T = -\overleftrightarrow{J}$$

$$\overleftrightarrow{M} \overleftrightarrow{J} \overleftrightarrow{M}^T = \left(\overleftrightarrow{\mathbb{1}} + \epsilon \overleftrightarrow{J} \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} \right) \left(\overleftrightarrow{J} \right) \left(\overleftrightarrow{\mathbb{1}} - \epsilon \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} \overleftrightarrow{J} \right)$$

$$= \overleftrightarrow{J} + \epsilon \overleftrightarrow{J} \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} \overleftrightarrow{J} - \overleftrightarrow{J} \epsilon \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} \overleftrightarrow{J}$$

$$= \overleftrightarrow{J}$$

ANY ICT SATISFIES THE SYMPLECTIC CONDITION

TWO CT FORMALISMS: GENERATOR FN OR SYMPLECTIC

CAN USE EITHER TO SHOW THAT CT HAVE THE FOUR PROPERTIES OF A "GROUP"

1) $\overleftrightarrow{\mathbb{1}}$

2) IF M IS CT, SO IS M^{-1}

3) IF $M = M_1 M_2$, M IS CT IF M_1 AND M_2 ARE CT

4) ASSOCIATIVE

$$M_1 (M_2 M_3) = (M_1 M_2) M_3$$

BACK TO ICT

$$\delta q_j = \epsilon \frac{\partial G}{\partial p_j} \qquad \delta p_j = -\epsilon \frac{\partial G}{\partial q_j}$$

CHOOSE $G = H$

$$\delta q_j = \epsilon \frac{\partial H}{\partial p_j} = \epsilon \dot{q}_j$$

$$\delta p_j = -\epsilon \frac{\partial H}{\partial q_j} = \epsilon \dot{p}_j$$

THIS TRANSFORMS

$$Q_i = q_i + \epsilon \dot{q}_i \qquad P_i = p_i + \epsilon \dot{p}_i$$