# POISSON BRACKETS

$$[u,v]_{pq} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

$$[u,v]_{pq} = \frac{\partial u^T}{\partial \vec{\eta}} \overleftrightarrow{J} \frac{\partial v}{\partial \vec{\eta}} \qquad \vec{\nabla}_{\eta} u = \frac{\partial u}{\partial \vec{\eta}}$$

$$\vec{\eta}^T = (q_1 \cdots q_n \, p_1 \cdots p_n) \qquad \overleftrightarrow{J} \begin{pmatrix} \overleftrightarrow{0} & \overleftrightarrow{1} \\ -\overleftrightarrow{1} & \overleftrightarrow{0} \end{pmatrix}$$

$$= \left(\frac{\partial u}{\partial q_1}, \frac{\partial u}{\partial q_2}, \frac{\partial u}{\partial p_1}, \frac{\partial u}{\partial p_2}\right) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial v / \partial q_1 \\ \partial v / \partial q_2 \\ \partial v / \partial p_1 \\ \partial v / \partial p_2 \end{pmatrix}$$

$$= -\frac{\partial u}{\partial p_1}\frac{\partial v}{\partial q_1} - \frac{\partial u}{\partial p_2}\frac{\partial v}{\partial q_2} + \frac{\partial u}{\partial q_1}\frac{\partial v}{\partial p_1} + \frac{\partial u}{\partial q_2}\frac{\partial v}{\partial p_2}$$

$$[q_j, q_k]_{qp} = 0 = [p_j, p_k]_{qp}$$

$$[q_j, p_k]_{qp} = \delta_{jk} = -[p_j, q_k]_{qp}$$

$$[\vec{\eta}, \vec{\eta}]_{\vec{\eta}} = \overleftrightarrow{J} \qquad [\vec{\eta}, \vec{\eta}]_{\vec{\eta}}\Big|_{lm} = [\eta_l, \eta_m]$$

NOW DO A CT $\vec{\xi}^T = (Q_1 \ldots Q_n \, P_1 \ldots P_n)$

$$[\vec{\xi}, \vec{\xi}]_{\vec{\eta}} = \frac{\partial \vec{\xi}^T}{\partial \vec{\eta}} \overleftrightarrow{J} \frac{\partial \vec{\xi}}{\partial \vec{\eta}} = \overleftrightarrow{M}^T \overleftrightarrow{J} \overleftrightarrow{M}$$

IF $\vec{\eta} \rightarrow \vec{\xi}$ IS A CT, THEN $MJM = J$

$$[\vec{\eta}, \vec{\eta}]_{\vec{\eta}} = \overleftrightarrow{J} \qquad [\vec{\xi}, \vec{\xi}]_{\vec{\eta}} = \overleftrightarrow{J}$$

IF CT, THEN $[\vec{\xi}, \vec{\xi}]_{\vec{\xi}} = \overleftrightarrow{J}$

PB OF THE CANONICAL VARIABLES $(q, p)$ ARE FUNDAMENTAL PB AND ARE INVARIANT UNDER CT

FUNDAMENTAL PB INVARIANT IFF SYMPECTIC CT

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NOW GENERALIZE $[u, v]_{\eta}$ $\quad M_{ij} = \frac{\partial \xi_i}{\partial \eta_j}$

$$\frac{\partial v}{\partial \vec{\eta}} = M^T \frac{\partial v}{\partial \vec{\xi}}$$

$$\frac{\partial v}{\partial \eta_i} = \frac{\partial v}{\partial \xi_j} \frac{\partial \xi_j}{\partial \eta_i} \Rightarrow M^T \frac{\partial v}{\partial \vec{\xi}}$$

$$\frac{\partial u^T}{\partial \vec{\eta}} = \left(M^T \frac{\partial u}{\partial \vec{\xi}}\right)^T = \frac{\partial u^T}{\partial \vec{\xi}} M$$

$$[u, v]_{\vec{\eta}} = \frac{\partial u^T}{\partial \vec{\eta}} \overleftrightarrow{J} \frac{\partial v}{\partial \vec{\eta}} = \frac{\partial u^T}{\partial \vec{\xi}} M^T J M \frac{\partial v}{\partial \vec{\xi}}$$

$$= \frac{\partial u^T}{\partial \vec{\xi}} \overleftrightarrow{J} \frac{\partial v}{\partial \vec{\xi}} = [u, v]_{\vec{\xi}}$$

ALL PB ARE CANONICAL INVARIANTS

WE KNOW HAM EOM INVARIANT UNDER CT
ANYTHING EXPRESSED IN PB IS INVARIANT UNDER CT
(ASSUMING $\lambda = 1$. NO SCALE XFORM)

PROPERTIES

$$[u,u] = 0$$

$$[u,v] = -[v,u] \quad \text{ANTISYMMETRIC}$$

$$[au + bv, w] = a[u,w] + b[v,w] \quad \text{LINEAR}$$

$$[uv, w] = [u,w]v + u[v,w]$$

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

LIE ALGEBRA JACOBI IDENTITY

a) VECTOR CROSS PRODUCTS $[A,B] = \vec{A} \times \vec{B}$

b) MATRIX COMMUTATORS $[A,B] = \overleftrightarrow{A}\overleftrightarrow{B} - \overleftrightarrow{B}\overleftrightarrow{A}$

$$\underset{\text{CLASSICAL}}{[u,v]} \rightarrow \frac{1}{i\hbar}\left(\hat{U}\hat{V} - \hat{V}\hat{U}\right)$$

PHASE SPACE INVARIANCE

$$(d\eta) = d^{2n}\eta = dq_1 \cdots dq_n\, dp_1 \cdots dp_n$$

$$(d\xi) = dQ_1 \cdots dQ_n\, dP_1 \cdots dP_n$$

$$\vec{\xi} = \overleftrightarrow{M}\,\vec{\eta} \qquad \|\overleftrightarrow{M}\| = \mathrm{abs}(\det(\overleftrightarrow{M}))$$

$$(d\xi) = \|\overleftrightarrow{M}\|\,(d\eta)$$

$$dQ\,dP = \begin{vmatrix} \partial q/\partial Q & \partial q/\partial P \\ \partial p/\partial Q & \partial p/\partial P \end{vmatrix} dq\,dp = [q,p]_{QP}\, dq\,dp$$

$$M^T J M = J$$

$$|M^T|\,|J|\,|M| = |J| \Rightarrow |M|^2 |J| = |J|$$

$$|M|^2 = 1$$

FOR A CT $\quad (d\xi) = (d\eta)$

IN 2D $\quad d\eta = dq\,dp$

$J = \int dq\,dp$ CONSERVED IN CT

EXAMPLE HO $\qquad [A,B] = \dfrac{\partial A}{\partial Q}\dfrac{\partial B}{\partial P} - \dfrac{\partial A}{\partial P}\dfrac{\partial B}{\partial Q}$

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q \qquad p = \sqrt{2m\omega P}\cos Q$$

$$[q,p]_{qp} = 1 \overset{?}{=} [q,p]_{QP} = \left[\sqrt{\frac{2P}{m\omega}}\sin Q,\ \sqrt{2m\omega P}\cos Q\right]_{QP}$$

$$= \sqrt{\frac{2P}{m\omega}}\cos Q\,\frac{1}{2}\sqrt{\frac{2m\omega}{P}}\cos Q - \frac{1}{2}\sqrt{\frac{2}{m\omega P}}\sin Q\left(\sqrt{2m\omega P}\sin Q\right)$$

$$= 1 \ \checkmark$$

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i}\dot{q}_i + \frac{\partial u}{\partial p_i}\dot{p}_i + \frac{\partial u}{\partial t}$$

$$= \frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t}$$

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

<OR> $\frac{du}{dt} = \frac{du^T}{d\vec{\eta}}\dot{\vec{\eta}} + \frac{\partial u}{\partial t} = \frac{\partial u^T}{\partial \vec{\eta}} J \frac{\partial H}{\partial \vec{\eta}} + \frac{\partial u}{\partial t}$

$$\dot{q}_i = [q_i, H] \qquad \dot{p}_i = [p_i, H] \qquad \dot{\vec{\eta}} = [\vec{\eta}, H]$$

$$\frac{dH}{dt} = \underbrace{\cancel{[H, H]}}_{\to 0} + \frac{\partial H}{\partial t}$$

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IF $u$ IS A CONSTANT OF THE MOTION

$$0 = \frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \Rightarrow [H, u] = \frac{\partial u}{\partial t}$$

IF $u \neq u(t)$ THEN

$u$ IS A CONSTANT OF THE MOTION IFF $[H, u] = 0$

IF $u, v$ ARE CONSTANTS OF THE MOTION THEN SO IS $[u, v]$

$$[H, [u, v]] + [u, \underbrace{\cancel{[v, H]}}_{\to 0}] + [v, \underbrace{\cancel{[H, u]}}_{\to 0}] = 0$$

GENERATE INFINITESIMAL CT (ICT) w/ PB

$$\vec{\xi} = \vec{\eta} + d\vec{\eta} \qquad d\vec{\eta} = \epsilon \overleftrightarrow{J} \frac{\partial G}{\partial \vec{\eta}}$$

REMINDER $F_2 = q_i P_i + \epsilon G(q, P, t)$ $\quad G = G(\vec{\eta})$

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j + \epsilon \frac{\partial G}{\partial q_j}$$

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \epsilon \frac{\partial G}{\partial P_j}$$

USE $[u, v]_\eta = \frac{\partial u}{\partial \vec{\eta}}^T \overleftrightarrow{J} \frac{\partial v}{\partial \vec{\eta}}$

$$[\vec{\eta}, u] = \overleftrightarrow{I} \overleftrightarrow{J} \frac{\partial u}{\partial \vec{\eta}} \qquad [\vec{\eta}, G] = \overleftrightarrow{J} \frac{\partial G}{\partial \vec{\eta}}$$

$$\delta \vec{\eta} = \epsilon [\vec{\eta}, G]$$

ICT in time $\epsilon \to dt$ $\quad G \to H$

$$\delta \vec{\eta} = dt [\vec{\eta}, H] = \dot{\vec{\eta}}\, dt = d\vec{\eta}$$

SYSTEM MOTION CONSISTS OF CONTINUOUS EVOLUTION USING ICT

H GENERATES SYSTEM MOTION

$\exists$ A CT FROM $p(t)\, q(t)$ TO $p(t_0)\, q(t_0)$

PASSIVE VIEW: SAME SYSTEM DIFF COORDS

ACTIVE VIEW: SYSTEM EVOLVES FROM $(q, p)$ TO $(Q, P)$

$$\partial u = u(B) - u(A)$$ WHERE AN ACTIVE ICT $A \to B$

$$\partial u = u(\vec{\eta} + \delta\vec{\eta}) - u(\vec{\eta})$$

$$= \frac{\partial u^T}{\partial \vec{\eta}} \delta\vec{\eta} = \epsilon \frac{\partial u^T}{\partial \vec{\eta}} \overleftrightarrow{J} \frac{\partial G}{\partial \vec{\eta}}$$

$$\partial u = \epsilon [u, G]$$

$$\partial \vec{\eta} = \epsilon [\vec{\eta}, G] = \delta\vec{\eta}$$

IF THE CT DEPENDS ON TIME, THEN $H = H(t)$

$H$ IS THE FN, IN A SPECIFIC PHASE, THAT DEFINES THE CANONICAL EOM

$$H(A) \to K(A')$$

$$\partial u = u(B) - u(A) \to \partial H = H(B) - K(A')$$