$$\partial u = u(B) - u(A)$$
$$= u(\vec{\eta} + \delta\vec{\eta}) - u(\vec{\eta})$$
$$= \epsilon\,[u, G]$$
$$\partial\vec{\eta} = \epsilon\,[\eta, G] = \delta\vec{\eta}$$

IF THE CT DEPENDS ON t, THEN SO DOES H

$$\partial H = H(B) - K(A')$$
$$K = H + \frac{\partial F}{\partial t}$$

ICT: $F_2 = q_i P_i + \epsilon G(q, P, t)$

$$K(A') = H(A') + \epsilon\frac{\partial G}{\partial t} = H(A) + \epsilon\frac{\partial G}{\partial t}$$
$$\partial H = H(B) - H(A) - \epsilon\frac{\partial G}{\partial t}$$
$$= \epsilon\,[H, G] - \epsilon\frac{\partial G}{\partial t} = -\epsilon\frac{dG}{dt}$$

If G IS A CONSTANT OF THE MOTION (c of m) THEN IT GENERATES AN ICT THAT LEAVES THE VALUE OF H UNCHANGED

$$\partial H = H(B) - H(A)$$

IF SYSTEM SYMMETRIC $\Rightarrow$ H UNCHANGED

ROTATIONS. H SYMMETRIC AROUND $\hat{z}$
ANYTHING THAT GENERATES (VIA ICT) ROTATIONS AROUND z IS CONSERVED

MORE GENERAL THAN CYCLIC VARIABLES

IF q_i IS CYCLIC, H INVARIANT UNDER ANY ICT THAT ONLY CHANGES q_i

$G(q,p) = p_i \qquad \delta p_j = -\epsilon \frac{\partial G}{\partial q_j} = 0$

$$\delta q_j = \epsilon \frac{\partial G}{p_j} = \epsilon \delta_{ij}$$

$\Rightarrow p_i$ IS A C OF M

MORE GENERALLY $\quad G_\ell = (\overleftrightarrow{J}\vec{\eta})_\ell = J_{\ell r}\eta_r$

$$\delta\eta_k = \epsilon J_{ks}\frac{\partial G_\ell}{\partial \eta_s} = \epsilon J_{ks}J_{\ell r}\delta_{rs} = \epsilon J_{ks}J_{\ell s}$$

$$= \epsilon(\overleftrightarrow{J}^{2})_{k\ell} = \epsilon\delta_{k\ell}$$

ROTATIONS: CARTESIAN COORDS ROTATE AROUND $\hat{z}$ BY θ

$$\delta x_i = -y_i d\theta \quad \delta y_i = x_i d\theta \quad \delta z_i = 0$$

$$\delta p_{x_i} = -p_{y_i} d\theta \quad \delta p_{y_i} = p_{x_i} d\theta \quad \delta p_{z_i} = 0$$

$$\delta p_j = -\epsilon \frac{\partial G}{\partial q_j} \quad \delta q_j = \epsilon \frac{\partial G}{\partial p_j}$$

$$G = x_i p_{y_i} - y_i p_{x_i} \qquad d\theta = \epsilon$$

$$G = L_z = (\vec{r}_i \times \vec{p}_i)_z$$

ACTIVE VIEW: ICT MOVES THE SYSTEM

FINITE CT $= \int$ ICT

$$U(0) \rightarrow U(\alpha)$$

$$\partial U = d\alpha\,[U, G] \qquad \frac{dU}{d\alpha} = [U, G]$$

$$U(\alpha) = U(0) + \alpha \left.\frac{dU}{d\alpha}\right|_0 + \frac{\alpha^2}{2!} \left.\frac{d^2U}{d\alpha^2}\right|_0 + \frac{\alpha^3}{3!} \left.\frac{d^3U}{d\alpha^3}\right|_0 + \ldots$$

$$\partial \frac{dU}{d\alpha} = d\alpha \left[\frac{dU}{d\alpha}, G\right] = d\alpha\,[[U, G], G]$$

$$\frac{d^2U}{d\alpha^2} = [[U, G], G] \qquad \frac{d^3U}{d\alpha^3} = [[[U, G], G], G]$$

$$U(\alpha) = U_0 + \alpha [U, G]_0 + \frac{\alpha^2}{2}[[U, G], G] + \frac{\alpha^3}{3}[\quad]$$

EXAMPLE: $G = L_z = x_i P_{y_i} - y_i P_{x_i}$

$$U = x_i \qquad [X_i, L_z] = \frac{\partial x_i}{\partial q_j}\frac{\partial L_z}{\partial P_j} - \frac{\partial X_i}{\partial p_j}\frac{\partial L_z}{\partial q_j}$$

$$[X_i, L_z]_0 = -y_i \qquad ([X_i, L_z], L_z) = -x_i$$

$$X_i = x_i - y_i\theta - x_i\frac{\theta^2}{2} + y_i\frac{\theta^3}{3!} + x_i\frac{\theta^4}{4!} - \ldots$$

$$= x_i\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) - y_i\left(\theta - \frac{\theta^3}{3!} + \ldots\right)$$

$$= x_i \cos\theta - y_i \sin\theta$$

EXAMPLE $G = H$, time

$$\frac{dU}{dt} = [U, H]$$

$$U(t) = U_0 + t[U, H] + \frac{t^2}{2!}[[U, H], H] + \ldots$$

$$H = \frac{p^2}{2m} - max$$

$$[x, H] = \frac{p}{m}$$

$$[[x, H], H] = \frac{1}{m}[p, H] = a$$

$$[[[x, H], H], H] = [a, H] = 0$$

$$x = x_0 + \frac{p_0 t}{m} + \frac{a}{2}t^2 + 0$$

SERIES METHOD SHOWS ICT $\Rightarrow$ CT

$\Rightarrow$ EOM

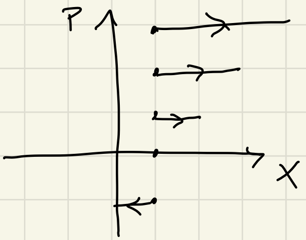
$u(t) = u e^{\hat{H}t}\Big|_0$ WHERE $\hat{H} = [\ , H]$

SIMILAR TO $e^{iHt}|\psi\rangle$

PHASE SPACE 2N DIMENSIONS

$N = 1$

1) FREE PARTICLE $\dot{x} = \frac{p}{m}$ $\dot{p} = 0$



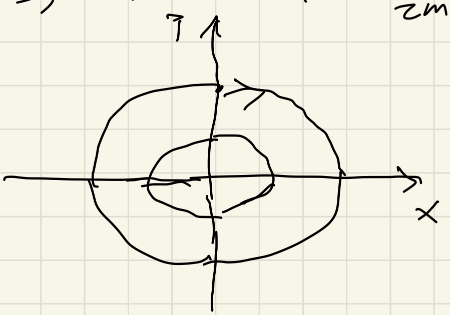
2) GRAVITY $H = \frac{p^2}{2m} + mgx$

$dx = \frac{p}{m} dt$ $dp = -mg\,dt$

SLOPE OF TRAJECTORY

$\text{"}\frac{dy}{dx}\text{"} = \frac{dp}{dx} = -m^2 g/p$

3) HO $H = \frac{p^2}{2m} + \frac{1}{2} k x^2$



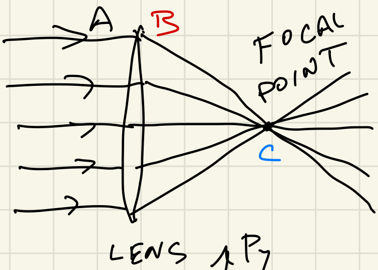
$dx = \frac{p}{m} dt$

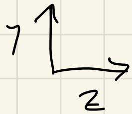
$dp = -kx$

CW OR CCW?

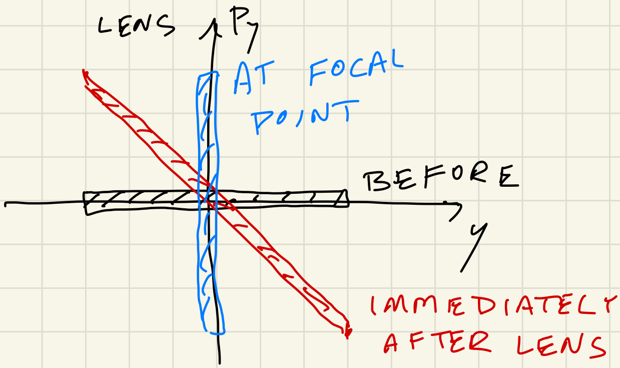
$p > 0 \Rightarrow x$ becoming more positive

4) LENS



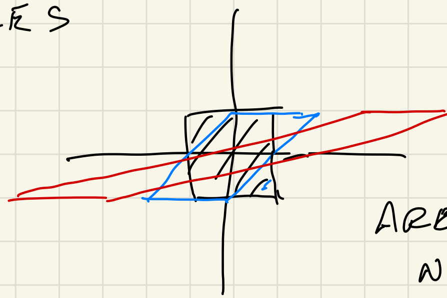


$$\Delta p_y = -ay$$



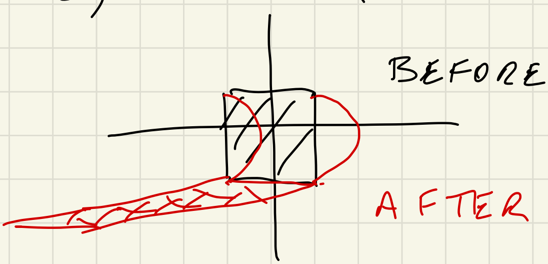
ENSEMBLES

1) FREE

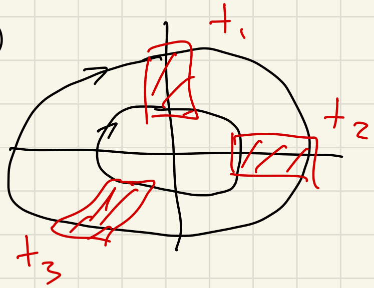


AREA = BASE × HT
NOT CHANGING

2) GRAVITY



HO



LIOUVILLE'S THM

1) N PARTICLES, SAME $H(\bar{q}, \bar{p}, t)$

2) DISTRIBUTED DENSELY IN SOME PHASE SPACE VOLUME V

SURFACE OF V HAS $2N-1$ DIMENSIONS

TRAJECTORIES CANNOT CROSS EACH OTHER

CANNOT CROSS THE SURFACE

$\therefore$ N IS CONSTANT

CAN USE A CT TO GO FROM A TO B

$V_{\text{PHASESPACE}} = \int dq_1 \cdots dq_n \, dp_1 \cdots dp_n$ IS CONSTANT

UNDER A CT $\Rightarrow$ V IS CONSTANT

$D = \dfrac{N}{V}$ CONSTANT