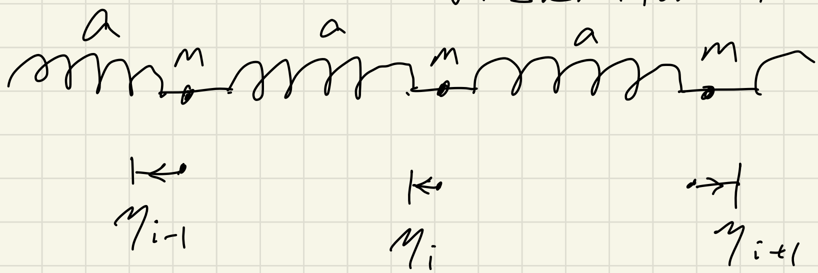
LONGITUDINAL VIBRATION IN ELASTIC ROD



η_i = displacement of mass i

$$T = \frac{1}{2}\sum_i m\dot{\eta}_i^2 \qquad V = \frac{1}{2}\sum_i k(\eta_{i+1}-\eta_i)^2$$

$$\mathcal{L} = T - V = \frac{1}{2}\sum_i a\left(\frac{m}{a}\dot{\eta}^2 - ka\left[\frac{\eta_{i+1}-\eta_i}{a}\right]^2\right)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}_i} - \frac{\partial \mathcal{L}}{\partial \eta_i} = 0$$

$$0 = \frac{m}{a}\ddot{\eta}_i - ka\left(\frac{\eta_{i+1}-\eta_i}{a^2}\right) + ka\left(\frac{\eta_i-\eta_{i-1}}{a^2}\right)$$

now look at $a \to 0$

$\frac{m}{a} = \mu$ spring constants $\sim k/a$

YOUNG'S MODULUS

rubber	~0.1 GPA
wood	~10 GPa
steel	200
diamond	1000

ENTIRE ROD $F = Y \frac{\Delta L}{L}$

STRESS STRAIN

(WE'RE IGNORING AREA OF ROD)

$$F = k(\eta_{i+1} - \eta_i) = \underbrace{ka}_{Y}\underbrace{\left(\frac{\eta_{i+1} - \eta_i}{a}\right)}_{\text{STRAIN}}$$

$$\frac{\eta_{i+1} - \eta_i}{a} \Rightarrow \frac{\eta(x+a) - \eta(x)}{a} = \left.\frac{d\eta}{dx}\right|_a$$

$\eta_i \to \eta(x)$ X REPLACES i

X IS NOT A COORDINATE

$$\mathcal{L} = \frac{1}{2}\int \left[\mu \dot{\eta}^2(x) - Y\left(\frac{d\eta(x)}{dx}\right)^2\right] dx$$

$$-ka\left(\frac{\eta_{i+1} - \eta_i}{a^2}\right) + ka\left(\frac{\eta_i - \eta_{i-1}}{a^2}\right)$$

$$= -Y\frac{1}{a}\left[\left.\frac{d\eta}{dx}\right|_x - \left.\frac{d\eta}{dx}\right|_{x-a}\right] = -Y\left.\frac{d^2\eta}{dx^2}\right|_x$$

EOM: $\mu\frac{d^2\eta}{dt^2} - Y\frac{d^2\eta}{dx^2} = 0$

WAVE EQ $v = \sqrt{Y/\mu}$

LAGRANGIAN $\mathcal{L} = \mathcal{L}\left(\eta, \frac{d\eta}{dx}, \frac{d\eta}{dt}, x, t\right)$

$$\delta I = \delta \int_1^2 \int \mathcal{L}\, dx\, dt$$

$\mathcal{L}$ = Lagrangian density (different dimensions)

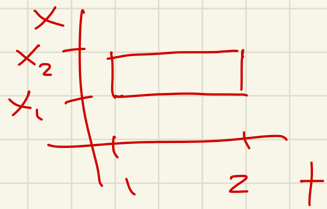
$\eta_i(t)$ $\dot{\eta}_i(t)$

FIXED AT t_1 AND t_2

NOW $\eta(x,t)$ $\dot{\eta}(x,t)$ FIXED AT t_1, t_2, x_1, x_2

$$\eta(x,t,\alpha) = \eta(x,t,0) + \alpha\, \xi(x,t)$$

$\xi(x,t)$ vanishes for $x = x_{lo}$ or x_{hi}
$t = t_1$ or t_2



$$\frac{dI}{d\alpha} = \int_{t_1}^{t_2} \int_{x_{lo}}^{x_{hi}} dx\, dt \left[\frac{\partial \mathcal{L}}{\partial \eta} \frac{\partial \eta}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{\eta}} \frac{\partial \dot{\eta}}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \eta'} \frac{\partial \eta'}{\partial \alpha} \right]$$

INTEGRATE BY PARTS $\eta' = \frac{\partial \eta}{\partial x}$

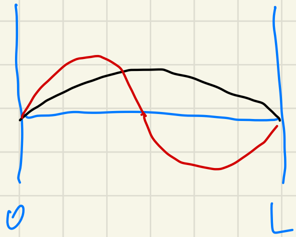
$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{\eta}} \frac{\partial \dot{\eta}}{\partial \alpha}\, dt = -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) \frac{\partial \eta}{\partial \alpha}\, dt$$

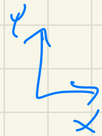
$$\int_{x_{lo}}^{x_{hi}} \frac{\partial \mathcal{L}}{\partial \eta'} \frac{\partial \eta'}{\partial \alpha}\, dx = -\int_{x_{lo}}^{x_{hi}} \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta'}\right) \frac{\partial \eta}{\partial \alpha}\, dx$$

$$\frac{dI}{d\alpha} = \int_{t_1}^{t_2} \int_{x_{lo}}^{x_{hi}} dx\, dt \left[\underbrace{\frac{\partial \mathcal{L}}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial \eta'}\right)}_{=0} \right] \left(\frac{\partial \eta}{\partial \alpha}\right) = 0$$

ELASTIC ROD $\frac{\partial \mathcal{L}}{\partial \dot{\eta}} = \mu \dot{\eta}$ $\frac{\partial \mathcal{L}}{\partial \eta'} = -Y \eta'$ $\frac{\partial \mathcal{L}}{\partial \eta} = 0 \Rightarrow$ EOM

TRANSVERSE WAVES





$$\frac{dy}{dx} = y' \ll 1$$

$$\mathcal{L} = \int_0^L dx \left(\frac{\lambda}{2}\dot{y}^2(x,t) - \frac{\tau}{2} y'^2(x,t) \right)$$

$\lambda = \frac{m}{L}$ $\quad \tau$ = TENSION

IF WE ADD GRAVITY $[-g\lambda y]$

$$y(x,t) = \sum_{k=1}^{\infty} a_k(t) \sin\frac{k\pi x}{L}$$

$a_k(t)$ GENERALIZED COORD

$$\int_0^L \frac{\lambda}{2}\dot{y}^2 dx = \sum_{k=1}^{\infty}\int_0^L \frac{\lambda}{2}\dot{a}_k^2(t)\underbrace{\sin^2\left(\frac{k\pi x}{L}\right)dx}_{\frac{1}{2}L}$$

$$= \frac{\lambda L}{4}\sum \dot{a}_k^2(t)$$

$$\int_0^L \frac{\tau}{2} y'^2 dx = \frac{\tau}{2}\sum\int_0^L \frac{k^2\pi^2}{L^2} a_k^2(t)\cos^2\frac{k\pi x}{L}dx$$

$$= \frac{L\tau}{4}\frac{\pi^2}{L^2}\sum k^2 a_k^2(t)$$

$$\mathcal{L} = \sum_{k=1}^{\infty}\left[\frac{\lambda L}{4}\dot{a}_k^2(t) - \frac{L\tau}{4}\frac{k^2\pi^2}{L^2}a_k^2(t)\right]$$

EOM $\quad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{a}_k} = \frac{L\lambda}{2}\ddot{a}_k$

$$\frac{\partial\mathcal{L}}{\partial a_k} = -\frac{L\tau}{2}\frac{k^2\pi^2}{L^2}a_k = \frac{L\lambda}{2}\ddot{a}_k$$

$a_k(t) = A_k\cos(\omega_k t + \phi_k) \qquad \omega_k^2 = \frac{\tau}{\lambda}\frac{k^2\pi^2}{L^2}$

REDO WITH y

$$\mathcal{L} = \frac{\lambda}{2}\dot{y}^2 - \frac{\tau}{2}y'^2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} + \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} - \frac{\partial \mathcal{L}}{\partial y} = 0$$

$$\lambda \ddot{y} - \tau y'' - 0 = 0$$

$$\lambda \frac{d^2y}{dt^2} = \tau \frac{d^2y}{dx^2}$$

$$y(x,t) = f(x-ct) + g(x+ct)$$

$$\lambda \ddot{y} - \tau y'' = \lambda c^2 \left[\ddot{f}(x-ct) + \ddot{g}(x+ct) \right] - \tau (f'' + g'')$$

$$\Rightarrow \quad c = \sqrt{\tau/\lambda}$$

$$y(x,t) = \int A(k) e^{i(\omega t + kx)} + B(k) e^{i(\omega t - kx)} \, dk$$

TWO DIMENSIONS

MEMBRANE CONSTANT DENSITY σ

$$T\,dA = \frac{1}{2}\sigma\, dA \left(\frac{du}{dt}\right)^2 \qquad u(x,y,t) \text{ VERTICAL DISPLACE}$$

$\tau(x,y)$ = SURFACE TENSION F/L

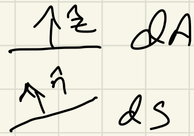
ENERGY DENSITY E/L^2

$$V\,dA = \tau\,(dS - dA)$$

dA ← UNSTRETCHED AREA

dS ← STRETCHED "

$$dA = \hat{z}\cdot\hat{n}\,dS$$



(dS small enough to ignore its curvature)

$$F(x,y,z,t) \equiv z - u(x,y,t) = 0$$

$$F = 0 \Rightarrow dF = \vec{\nabla}F\cdot d\vec{r} = 0$$

$d\vec{r}$ is ALONG THE SURFACE SO
$\vec{\nabla}F \perp d\vec{r}$ AND $\vec{\nabla}F \parallel \hat{n}$

$$\hat{n} = \frac{\vec{\nabla}F}{|\vec{\nabla}F|} = \frac{\vec{\nabla}F}{\left[(\vec{\nabla}F)^2\right]^{1/2}} = \frac{\hat{z} - \vec{\nabla}u}{\left[1 + (\vec{\nabla}u)^2\right]^{1/2}}$$

$$\hat{z}\cdot\hat{n} = \frac{1}{\left[1+(\vec{\nabla}u)^2\right]^{1/2}}$$

$$V\,dA = \tau\,(dS - dA)$$
$$= \tau\,dA\left(\left[1+(\vec{\nabla}u)^2\right]^{1/2} - 1\right)$$

TO 1ST ORDER

$$= \frac{1}{2}\tau\,dA\,(\vec{\nabla}u)^2$$

$$\mathcal{L} = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(\vec{\nabla}u)^2 \qquad L = \int \mathcal{L}\,dA$$

EOM $\Rightarrow$ $\sigma\ddot{u} = \tau\nabla^2 u$

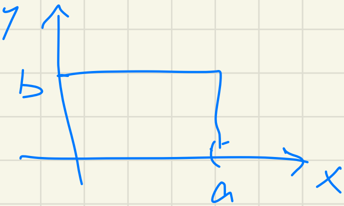
$$\frac{1}{c^2}\ddot{u} = \nabla^2 u \qquad c^2 = \tau/\sigma$$

NORMAL MODE SOLUTIONS

$$u(\vec{x},t) = \rho(\vec{x})\cos(\omega t + \phi)$$

$$(\nabla^2 + k^2)\rho(x,y) = 0 \qquad k = \omega/c$$

RECTANGULAR CLAMPED BOUNDARIES



$$\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + k^2 \rho = 0$$

$$\rho(x,y) = X(x)\,Y(y)$$

$$k^2 + \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$$

X, Y INDEPENDENT

⇒ BOTH SIDES CONSTANT

k_y^2

$$\frac{d^2X}{dx^2} = -k_x^2 X \qquad \frac{d^2Y}{dy^2} = -k_y^2 Y$$

$$k_x^2 + k_y^2 = k^2$$

$$X(x) = A_m\sqrt{\frac{2}{a}}\sin k_x x \qquad Y(y) = B_n\sqrt{\frac{2}{b}}\sin k_y y$$

$$k_x = \frac{m\pi}{a} \qquad k_y = \frac{n\pi}{b}$$

$$\psi_{mn}(x,y) = \sqrt{\frac{4}{ab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \cdot A_m B_n$$

$$\frac{w_{mn}^2}{c^2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = (k_x^2 + k_y^2)$$