$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\overleftrightarrow{L} = \begin{pmatrix} \gamma & -\gamma\vec{\beta}^T \\ -\gamma\vec{\beta} & \overleftrightarrow{A} \end{pmatrix}$$

OUTER PRODUCT

$$\overleftrightarrow{A} = \underline{1} + (\gamma - 1)\frac{\vec{\beta}\,\vec{\beta}^T}{\beta^2}$$

$$A_{ij} = \delta_{ij} + (\gamma - 1)\frac{\beta_i \beta_j}{\beta^2}$$

$$(\Delta s)^2 = (\Delta ct)^2 - (\Delta\vec{r})^2$$

LORENTZ INVARIANT

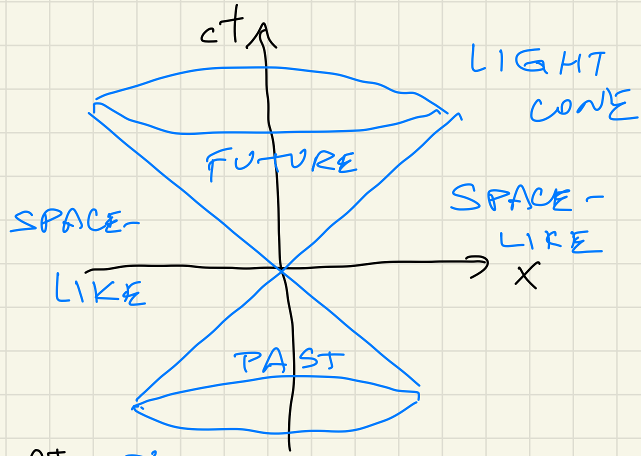
$$ct' = \gamma ct - \beta\gamma x$$
$$x' = -\beta\gamma ct + \gamma x$$

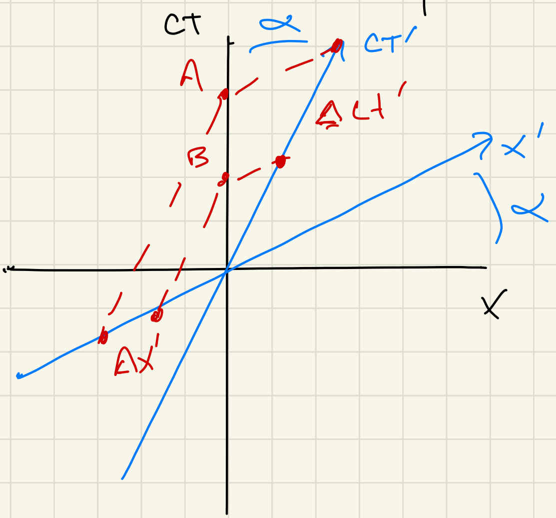
$$(ct')^2 - x'^2 = \gamma^2c^2t^2 - \cancel{2\beta\gamma^2 ctx} + \beta^2\gamma^2x^2$$
$$- \beta^2\gamma^2c^2t^2 + \cancel{2\beta\gamma^2 ctx} - \gamma^2x^2$$
$$= c^2t^2(\gamma^2 - \beta^2\gamma^2) - x^2(\gamma^2 - \beta^2\gamma^2)$$
$$\gamma^2 - \beta^2\gamma^2 = \gamma^2(1-\beta^2) = 1 \checkmark$$

$(\Delta s)^2 > 0$ time like separated — 2 EVENTS NEVER SIMULTANEOUS; $\exists$ FRAME WHERE 2 EVENTS AT SAME $\vec{r}$

$(\Delta s)^2 = 0$ light like

$(\Delta s)^2 < 0$ space like — 2 EVENTS CAN BE SIMUL BUT NEVER AT SAME $\vec{r}$




$$ct' = \gamma ct - \beta\gamma x$$

$x'$ axis $\Rightarrow ct' = 0$

$$\gamma ct = \beta\gamma x$$

$$x' = -\beta\gamma ct + \gamma x$$

$ct'$ axis $\Rightarrow x' = 0$

$$x = \beta ct$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{0.8} = \frac{5}{4}$$

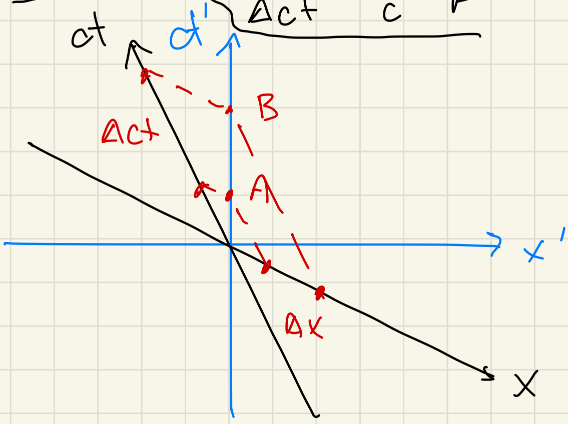
$$\Delta ct' = \gamma \Delta ct$$

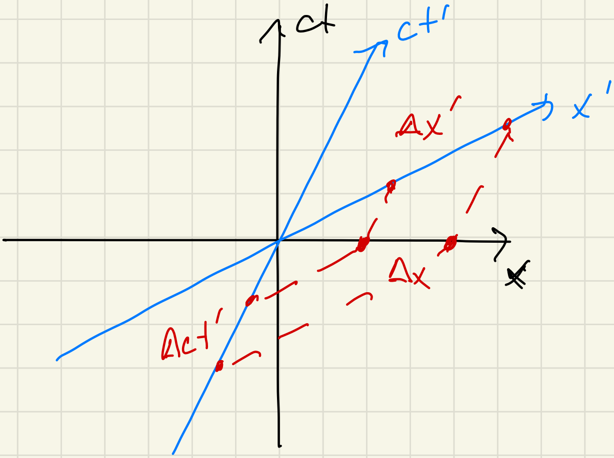
$$\Delta x' = -\beta\gamma\Delta ct + \cancel{\gamma\Delta x}$$

$$= -\tfrac{3}{4}\Delta ct$$

$\beta = 0.6$

$$\tan\alpha = \frac{\Delta x}{\Delta ct} = \frac{v}{c} = \beta$$






# VELOCITY ADDITION

$\overleftrightarrow{L} = \overleftrightarrow{R}\,\overleftrightarrow{L}_0$ = ROTATION $\otimes$ PURE BOOST

R NOT SYMMETRIC
$L_0$ IS "
$\Rightarrow$ L IS NOT "

$\overleftrightarrow{L} = \overleftrightarrow{L}_0'\,\overleftrightarrow{R}'$ (DIFFERENT MATRICES)

$\overleftrightarrow{L}\,\overleftrightarrow{L}^{-1} = \mathbb{1}$ $\Rightarrow$ 10 CONSTRAINTS

$\Rightarrow$ 6 DOF $\quad \vec{\beta}$ 3
$\quad \vec{R}$ 3

# PARALLEL BOOSTS

$S_2$ MOVING w/ $V$ WRT $S_1$
$S_3$ " w/ $V'$ WRT $S_2$

$$\overleftrightarrow{L}_{1\text{-}3} = \overleftrightarrow{L}_{23}\overleftrightarrow{L}_{12} = \begin{bmatrix} \gamma' & -\beta'\gamma' & 0 & 0 \\ -\beta'\gamma' & \gamma' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} \gamma\gamma'(1+\beta\beta') & -\gamma\gamma'(\beta+\beta') & 0 & 0 \\ -\gamma\gamma'(\beta+\beta') & \gamma\gamma'(1+\beta\beta') & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\beta'' = \frac{\beta+\beta'}{1+\beta\beta'}$$

NON PARALLEL BOOSTS (THOMAS PRECESION)

$L_1 L_2 = L_3 \quad L_3 \in$ LORENTZ GROUP

$L_3$ WILL HAVE BOOST $\otimes$ ROTATION

$\overleftrightarrow{L}_1$ : $\beta_1$ ALONG X

$\overleftrightarrow{L}_2$ : $\beta_2 \ll \beta_1$ IN X-Y PLANE

$\overleftrightarrow{L}_3 = \overleftrightarrow{L}_2\overleftrightarrow{L}_1$ NOT SYMMETRIC

$\therefore$ IT HAS A ROTATION

$$\beta_{3x} = \beta_1 \quad \beta_{3y} = \frac{\beta_{2y}}{\gamma_1} \qquad \beta_3^2 = \beta_1^2$$

$$\gamma_3 = \gamma_1$$

$\Rightarrow$ ROTATION AROUND Z $\quad \Delta\Omega = \beta_{3y}\beta_1\dfrac{\gamma_1 - 1}{\beta_1^2}$

ROTATION AROUND $z$ $\quad \Delta\Omega = \beta_{3y}\,\beta\,\dfrac{\gamma-1}{\beta^2}$ $\quad \begin{pmatrix} \beta_1 = \beta \\ \gamma_1 = \gamma \end{pmatrix}$

PARTICLE MOVING W/ NONCONSTANT $\vec{v}$

$S_1$ = LAB FRAME

$S_2$ = PARTICLE FRAME AT $t = t_2$ $\quad \vec{v} = \beta c \hat{x}$

$S_3$ = " " AT $t = t_3 = t_2 + \Delta t$

$$\Delta\vec{v} = \beta_{2y}\, c\, \hat{y}$$

$$\Delta\vec{\Omega} = -(\gamma-1)\,\frac{\vec{v}\times\Delta\vec{v}}{v^2}$$

SPIN DIRECTION WILL PRECESS W/

$$\vec{\omega} = \frac{d\vec{\Omega}}{dt} = -(\gamma-1)\,\frac{\vec{v}\times\vec{a}}{v^2}$$

IF $\beta \ll c$ THEN $\gamma \simeq 1 + \frac{1}{2}\beta^2$

$$\vec{\omega} = \frac{1}{2c^2}\,\vec{a}\times\vec{v}$$ OBSERVED IN LAB!

# MINKOWSKI SPACE

4-VECTOR $V^\mu$ $\mu = 0,1,2,3$ $(V^k \quad k = 1,2,3)$

$x^\mu = (ct, x, y, z) = (ct, \vec{r})$ SPACELIKE OR TIMELIKE

$p^\mu = (E/c, \vec{p}) = m_0 c(\gamma, \gamma\vec{\beta})$ TIMELIKE FOR REAL PARTICLES

$v^\mu = (\gamma c, \gamma \vec{v})$ TL

$F^\mu = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma \frac{d\vec{p}}{dt}\right)$ $\quad (F^\mu)^2 = -(F_{NEWTON})^2$

SL

CURRENT DENSITY $J^\mu = (\gamma \rho c, \gamma \vec{J})$ $\quad (J^\mu)^2 = \rho^2 c^2$ TL

SCALAR PRODUCT $g(x^\mu, y^\nu) = s$

SCALARS INVARIANT UNDER ROTATION AND BOOSTS

$$g(u,v) = g(v,u) = u \cdot v$$

$$= u_0 v_0 - \vec{u} \cdot \vec{v}$$

$$= u_0 v_0 - u_1 v_1 - u_2 v_2 - u_3 v_3$$

SUPERSCRIPTS VS SUBSCRIPTS NOW MATTER

$$g(p,p) = \frac{E^2}{c^2} - \vec{p}^2 = m_0^2 c^2$$

$$g(dx, dx) = (ds)^2 = (dct)^2 - (d\vec{r})^2$$

$g(v,v) = 0$ LIGHTLIKE $\quad > 0$ TIMELIKE

$< 0$ SPACELIKE

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

SUMMATION CONVENTION ONLY APPLIES TO $\begin{smallmatrix}\alpha \\ \alpha\end{smallmatrix}$ OR $\begin{smallmatrix}\beta \\ \beta\end{smallmatrix}$

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$U \cdot V = U^\alpha V^\beta g_{\alpha\beta} = U^0V^0 - U^1V^1 - U^2V^2 - U^3V^3$$

$$g_\alpha{}^\beta = \mathbb{1} \quad \text{SINCE} \quad g_\nu{}^\mu x^\nu = x^\mu$$

$$g^{\alpha\beta} = g_{\alpha\beta}$$

| SYSTEM | VECTORS (CONTRAVARIANT) | 1-FORMS (COVARIANT) | METRIC |
|---|---|---|---|
| EUCLIDEAN | $dx, dy, dz$ | SAME | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| SPHERICAL | $dr, d\theta, d\phi$ | $(dr, r^2 d\theta, r^2 \sin^2\theta\, d\phi)$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$ |
| QUANTUM | $\lvert i \rangle$ | $\langle j \rvert$ | $\lvert i \rangle\langle j \rvert$ |
| MINKOWSKI | $(c\,dt, d\vec{r})$ | $(c\,dt, -d\vec{r})$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ |

INNER PRODUCT

$$x^\alpha = g^{\alpha\beta} x_\beta$$

$$x^\alpha x_\alpha = (\text{VECTOR})(\text{1-FORM})$$