\[ L = -mc^2 \sqrt{1 - \beta^2} - V \]

Start with the Lagrangian
\[ L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \dot{\theta}^2 \cdot \hat{A} - 9\phi \]

Not to be Lorentz invariant.

Total dynamical energy \( \delta mc^2 \)

Elem use \( \frac{d}{dt} \) want \( \frac{d}{d\gamma} \)

\[ \Rightarrow \text{divide by } \gamma = (1 - \frac{\dot{x}^2}{c^2})^{1/2} \]

Subtract the rest energy
\[ \frac{\delta mc^2 - mc^2}{\gamma} = mc^2 - mc^2 (1 - \beta^2)^{1/2} \]

Neglect constant term \( mc^2 \)

\[ \Rightarrow -mc^2 (1 - \beta^2)^{1/2} \]
**Covariant Formulation**

**Ideally** \( \mathbf{r} \) is an independent coord

\[ U \cdot U = U^\mu U_\mu = c^2 \]

The set of \( \{ U^\mu \} \) are not independent

Ignore this constraint and apply it later

\[ x''^\mu = \frac{d x'^\mu}{d \tau} \]

\[ \delta I = \delta \int_{\tau_1}^{\tau_2} \Lambda(x'^\mu, x''^\mu) d\tau \]

Want \( \Lambda \) that gives us the EOM

\[ I = \int_{\tau_1}^{\tau_2} L(x^i, \dot{x}^i) d\tau = \frac{1}{c} \int_{\tau_1}^{\tau_2} \Lambda(x^i, c \frac{\dot{x}^i}{x^0}) x^i_0 d\tau \]

\[ \Lambda(x'^\mu, x''^\mu) = \frac{x^0}{c} L(x'^\mu, c \frac{x''^i}{x^0}) \]

\[ \Lambda(x'^\mu, a x''^\mu) = a \Lambda(x'^\mu, x''^\mu) \]

\[ \Lambda = x''^\mu \frac{\partial \Lambda}{\partial x''^\mu} \]

\[ \frac{\partial}{\partial \tau} \left( \frac{\partial \Lambda}{\partial U^\mu} \right) - \frac{\partial \Lambda}{\partial x^\mu} = 0 \]

\[ U^\mu = \frac{d x'^\mu}{d \tau} \]

One possible free particle \( \Lambda \)

\[ \Lambda = \frac{1}{2} m U^\mu U_\mu \]

\[ \Lambda = \frac{1}{2} m U^\mu U_\mu + g U^\mu A_\mu(x^i) \]
\[
\frac{d}{dt} (\mu \nu) = -g \frac{\partial A^\mu}{\partial t} + \frac{2}{\partial x^\nu} \left( g \mu \nu A^\nu \right)
\]

This is what we got before

\[
\frac{\partial \mu \nu}{\partial t} = K \text{ Minkowski force}
\]

\[
K^\mu = -g \left( 2 (\mu \nu A^\nu) - \frac{\partial A^\mu}{\partial x^\nu} \right)
\]

\[
F_i = K \sqrt{1 - \beta^2}
\]

**Mechanical Momentum** \( \mu \nu = \mu \nu \)

**Canonical Momentum**

\[
\mu \nu = g^{\mu \nu} \frac{\partial \lambda}{\partial \nu} = \frac{2 \lambda}{\partial \nu} = \mu \nu + gA^\nu
\]

\[
\mu = \frac{E}{c} + g \frac{\phi}{c} = \frac{1}{c} E \text{ Total Energy}
\]

\[
\hat{\mu} = \hat{\mu} + gA
\]

\[
E^2 = c^2 (\hat{\mu} - gA)^2 + m^2 c^4
\]

Does not work for internal forces potentials action at a distance problems due to lack of retarded pot.
RELATIVISTIC HAMILTONIANS

Non covariant, works in a specific Lorentz frame. Time as a parameter, not a coordinate.

\[ L = -mc^2 \sqrt{1 - \beta^2} - V \]

\[ H = T + V \quad \text{total energy} \]

\[ T^2 = p^2 c^2 + m^2 c^4 \quad (T \text{ is now mass energy plus KE}) \]

\[ H = \sqrt{p^2 c^2 + m^2 c^4} + V \]

Electric LAGRANGIAN \[ L = -mc^2 \sqrt{1 - \beta^2} + q A \cdot \dot{v} - q \phi \]

\[ \Rightarrow H = T + q \phi \]

\[ P^i = mu^i + q A^i \quad \text{canonical momentum} \]

\[ T^2 = (\sqrt{p} - q A)^2 c^2 + m^2 c^4 \]

\[ H = \sqrt{(\sqrt{p} - q A)^2 c^2 + m^2 c^4} + q \phi \]

\[ \frac{H - q \phi}{c^2} = P_0 \quad \text{zeroth component of} \quad m u^0 + q A^0 \]

\[ H \text{ is not invariant. It is not a Lorentz scalar. It transforms like the 0th comp. of a 4-vector} \]

\[ \text{We still have} \quad \frac{\partial H}{\partial p_j} = \dot{p}_j, \quad \frac{\partial H}{\partial q_j} = \dot{q}_j \]
COVARIANT APPROACH

$x^\prime$ is a coordinate. It has a conjugate momentum.

Assume $f \in \Theta(q)$ which is monotonic in $q$

\[
g_i = \frac{\partial g_i}{\partial \theta} \Rightarrow \Lambda(q, g', t, t') = t' \Lambda(q, q', t, t')
\]

\[
g, q' \text{ are } \exists q, q' \text{ such that}
\]

**Canonical momentum conjugate to $t$**

(Use \( \dot{q}_i = q_i' = \frac{2q_i}{\partial \theta} \frac{\partial g_i}{\partial \theta} \))

\[
P_t = \frac{\partial \Lambda}{\partial g'} = 1 + t' \frac{\partial g_l}{\partial \theta} = 1 - 9 \cdot \frac{\partial g_l}{\partial t'} = -H
\]

\[
P_i = \frac{\partial \Lambda}{\partial \dot{q}_i} = t' \frac{\partial g_l}{\partial \theta} = t' \left( \frac{\partial g_l}{\partial \theta} \frac{1}{t'} \right) = P_i \checkmark
\]

Other momenta unchanged

But the covariant $\Lambda$ we used is 1st order homogenous in $q$,

\[
\Rightarrow H = 0
\]

Try $\Lambda(x', u) = \frac{1}{2} m u u$ (free)

This gives correct form

\[
H = \dot{q} P - L = P^\mu P^\nu / 2m
\]
\( E + m = \sqrt{(x^\mu, u^\mu)} = \frac{1}{2} m u^\mu + g u^\mu A_\mu (x^\mu) \)

\[
P_\nu = m u_\nu + g A_\nu
\]

\[
H = \frac{(P_\nu - g A_\nu)(P^\nu - g A^\nu)}{2 m}
\]

8 eq of motion
6 independent

\[
\frac{dx^\mu}{\sqrt{-g}} = \frac{\partial H}{\partial p^\mu}
\]

\[
\frac{dp^\mu}{\sqrt{-g}} = -\frac{\partial H}{\partial x^\mu}
\]

\[
U^0 = \frac{\partial H}{\partial p^0} = \frac{1}{m} (P^0 - g A^0)
\]

\[
P^0 = \frac{1}{c} (T + g \phi) = \frac{H}{c}
\]

\[
\frac{dp^0}{\sqrt{1 - \beta^2}} = -\frac{1}{c} \frac{\partial H}{\partial \gamma}
\]

\[
\frac{dH}{\gamma} = \sqrt{1 - \beta^2} \frac{\partial H}{\gamma}
\]

REALLY TOUGH TO DO WITH OTHER POTENTIALS
EVEN TOUGHER WITH INTERNAL (PARTICLE-PARTICLE) POTENTIALS