$$\mathcal{L} = -mc^2\sqrt{1-\beta^2} - V$$

START WITH NR LAGRANGIAN

$$\mathcal{L} = \underbrace{\frac{1}{2}m\dot{\vec{x}}^2}_{\text{NOT LI}} + \underbrace{q\dot{\vec{x}}\cdot\vec{A} - q\phi}_{\text{LORENTZ INV}}$$

TOTAL DYNAMICAL ENERGY γmc^2

ELEM USE $\frac{d}{dt}$, WANT $\frac{d}{d\tau}$

$\Rightarrow$ DIVIDE BY $\gamma = \left(1-\dot{\vec{x}}^2/c^2\right)^{-1/2}$

SUBTRACT THE REST ENERGY

$$\frac{\gamma mc^2 - mc^2}{\gamma} = mc^2 - mc^2\left(1-\beta^2\right)^{1/2}$$

NEGLECT CONSTANT TERM mc^2

$$\Rightarrow -mc^2\left(1-\beta^2\right)^{1/2}$$

COVARIANT FORMULATION

IDEALLY τ IS AN INDEPENDENT ~~COORD~~ PARAMETER

$U \cdot U = U^\mu U_\mu = c^2$ THE SET OF $\{U^\nu\}$ ARE NOT INDEPENDENT

IGNORE THIS CONSTRAINT AND APPLY IT LATER

$$x'^\mu = \frac{dx^\mu}{d\tau} \qquad \dot{x}^\mu = \frac{dx^\mu}{dt}$$

$$\delta I = \delta \int_{\tau_1}^{\tau_2} \Lambda(x^\mu, x'^\mu)\, d\tau$$

WANT Λ THAT GIVES US THE EOM

$$I = \int_{t_1}^{t_2} \mathcal{L}(x^j, t, \dot{x}^j)\, dt = \frac{1}{c}\int_{\tau_1}^{\tau_2} \mathcal{L}\left(x^j, c\frac{x'^j}{x'^0}\right) x'^0\, d\tau$$

$$\Lambda(x^\mu, x'^\mu) = \frac{x'^0}{c}\, \mathcal{L}\left(x^\mu, c\frac{x'^j}{x'^0}\right)$$

$$\Lambda(x^\mu, a x'^\mu) = a\, \Lambda(x^\mu, x'^\mu)$$

$$\Lambda = x'^\mu \frac{\partial \Lambda}{\partial x'^\mu}$$

$$\frac{d}{d\tau}\left(\frac{\partial \Lambda}{\partial U^\nu}\right) - \frac{\partial \Lambda}{\partial x^\nu} = 0 \qquad U^\nu = \frac{dx^\nu}{d\tau}$$

ONE POSSIBLE FREE PARTICLE Λ

$$\Lambda = \tfrac{1}{2} m U_\nu U^\nu$$

$$\Lambda = \tfrac{1}{2} m U_\nu U^\nu + q U^\mu A_\mu(x^\lambda)$$

$$\frac{d}{dt}(mu^\nu) = -q\frac{dA^\nu}{dt} + \frac{\partial}{\partial x_\nu}(q u^\mu A_\mu)$$

THIS IS WHAT WE GOT BEFORE

$$\frac{dp^\mu}{d\tau} = K^\mu$$ MINKOWSKI FORCE

$$K_\mu = -q\left(\frac{\partial(u_\nu A^\nu)}{\partial x^\mu} - \frac{dA_\mu}{d\tau}\right)$$

$$F_i = K_i\sqrt{1-\beta^2}$$

MECHANICAL MOMENTUM $p^\mu = mu^\mu$

CANONICAL MOMENTUM

$$P^\mu = g^{\mu\nu}\frac{\partial\Lambda}{\partial u^\nu} = \frac{\partial\Lambda}{\partial u_\nu} = mu^\mu + qA^\mu$$
$$= p^\mu + qA^\mu$$

$$P^0 = \frac{E}{c} + q\frac{\phi}{c} = \frac{1}{c}\mathcal{E}$$ TOTAL ENERGY

MECHANICAL ENERGY

$$\vec{P} = \vec{p} + q\vec{A}$$

$$E^2 = c^2(\vec{P} - q\vec{A})^2 + m^2c^4$$

DOES NOT WORK FOR INTERNAL ~~FORCES~~ POTENTIALS

ACTION AT A DISTANCE PROBLEMS DUE TO LACK OF RETARDED POT.

RELATIVISTIC HAMILTONIANS

NON COVARIANT, WORKS IN A SPECIFIC LORENTZ FRAME

TIME AS A PARAMETER, NOT A COORDINATE

$$\mathcal{L} = -mc^2\sqrt{1-\beta^2} - V$$

$$H = T + V \quad \text{TOTAL ENERGY}$$

$$T^2 = p^2c^2 + m^2c^4$$ (T IS NOW MASS ENERGY PLUS KE)

$$H = \sqrt{p^2c^2 + m^2c^4} + V$$

EM LAGRANGIAN $\mathcal{L} = -mc^2\sqrt{1-\beta^2} + q\vec{A}\cdot\vec{v} - q\phi$

$$\Rightarrow \quad H = T + q\phi$$

$$P^i = mu^i + qA^i$$ CANONICAL MOMENTUM

$$T^2 = (\vec{P} - q\vec{A})^2c^2 + m^2c^4$$

$$H = \left[(\vec{P} - q\vec{A})^2c^2 + m^2c^4\right]^{1/2} + q\phi$$

$$\frac{H - q\phi}{c} = P^0$$ zeroth component of $mu^\nu + qA^\nu$

H IS NOT INVARIANT. IT IS NOT A LORENTZ SCALAR. IT TRANSFORMS LIKE THE 0^{TH} COMP. OF A 4-VECTOR

WE STILL HAVE $-\dfrac{\partial H}{\partial q_j} = \dot{P}_j \qquad \dfrac{\partial H}{\partial P_j} = \dot{q}_j$

COVARIANT APPROACH

x^0 IS A COORDINATE. IT HAS A CONJUGATE MOMENTUM

ASSUME $\exists\ \theta(\tau)$ WHICH IS MONOTONIC IN τ

$$q_i' = \frac{\partial q_i}{\partial \theta}$$

$$\Rightarrow \Lambda(q, q', t, t') = t' \mathcal{L}\left(q, \frac{q'}{t'}, t\right)$$

q, q' ARE $\{q_i\}\ \{q_i'\}$

CANONICAL MOMENTUM CONJUGATE TO t

(USE $\dot{q}_i = \frac{q_i'}{t'} = \frac{\partial q_i}{\partial \theta} \Big/ \frac{\partial t}{\partial \theta}$)

$$P_t = \frac{\partial \Lambda}{\partial t'} = \mathcal{L} + t' \frac{\partial \mathcal{L}}{\partial t'}$$

$$= \mathcal{L} - \frac{q_i'}{t'} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \mathcal{L} - \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \underline{-H}$$

$$P_i = \frac{\partial \Lambda}{\partial q_i'} = t' \frac{\partial \mathcal{L}}{\partial q_i'} = t' \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{1}{t'} \right) = P_i \checkmark$$

OTHER MOMENTA UNCHANGED

BUT THE COVARIANT $\mathcal{L}$ WE USED IS 1^{st} ORDER HOMOGENOUS IN q'

$$\Rightarrow H = 0$$

TRY $\Lambda(x^\mu, u^\mu) = \frac{1}{2} m u_\mu u^\mu$ (FREE)

THIS GIVES CORRECT EOM

$$H = \dot{q} p - \mathcal{L} = p_\mu p^\mu / 2m$$

E+M: $\Lambda(x^\mu, u^\mu) = \frac{1}{2} m u_\nu u^\nu + q u^\nu A_\nu(x^\lambda)$

$$P_\nu = m u_\nu + q A_\nu$$

$$H = \frac{(P_\nu - qA_\nu)(P^\nu - qA^\nu)}{2m}$$

8 EQ OF MOTION
6 INDEPENDENT

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p^\nu} g^{\mu\nu} \qquad \frac{dp^\mu}{d\tau} = -\frac{\partial H}{\partial x^\nu} g^{\mu\nu}$$

$$u^0 = \frac{\partial H}{\partial p^0} = \frac{1}{m}(p^0 - qA^0)$$

$$p^0 = \frac{1}{c}(T + q\phi) = \frac{H}{c}$$

$$\frac{dp^0}{d\tau} = \frac{1}{\sqrt{1-\beta^2}} \frac{dp^0}{dt} = -\frac{1}{c} \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sqrt{1-\beta^2} \frac{\partial H}{\partial t}$$

REALLY TOUGH TO DO WITH OTHER POTENTIALS

EVEN TOUGHER WITH INTERNAL (PARTICLE-PARTICLE) POTENTIALS