CONSTRAINTS

HOLONOMIC $f(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_n, t) = 0$ no velocities

NON HOLONOMIC

TIME ~~dep~~ VS NON TIME DEP

HOLONOMIC → GENERALIZED COORDS
<OR> LAGRANGE MULT
NON HOLONOMIC →

GENERALIZED COORDS $\{\vec{r}_i, \vec{v}_i\} \Rightarrow \{q_i, \dot{q}_i\}$

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t}$$

GENERALIZED FORCE $Q_j = \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

IF $\vec{F}_i = -\vec{\nabla}_i V$ THEN $Q_j = -\frac{\partial V}{\partial q_j}$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \qquad \mathcal{L} = T - V$$

VELOCITY-DEPENDENT POTENTIALS

IF $\exists\ U(q_j, \dot{q}_j) \ni$

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j}$$

THEN USE $\mathcal{L} = T - U$

E + M $\quad U = q\phi - q\vec{A}\cdot\vec{v}$

V-DEPENDENT FRICTION $\quad F_x = -k_x V_x$

$\Rightarrow$ DISSIPATION FN

2] LEAST ACTION

$$\delta I = \delta \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t)\, dt = 0$$

EX: MINIMIZING SURFACES
MINIMUM TIME (BRACHISTOCHRONE)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

ASSUMES $\{q_i\}$, $\{\dot{q}_i\}$ ARE INDEPENDENT

$\Rightarrow$ CONSTRAINTS HOLONOMIC

ASSUMES FORCES DERIVEABLE FROM SCALAR $V(q_i, \dot{q}_i, t)$

N VARIABLES, m CONSTRAINTS

$$I = \delta \int_{t_1}^{t_2} \left(\mathcal{L} + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \right) dt$$

$$f_\alpha = f_\alpha(\{q_i\}\ \{\dot q_i\}\ t) = 0$$

f_α CONSTRAINT EQ

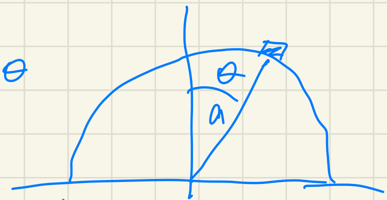
λ_α LAGRANGE MULT

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_k} - \frac{\partial \mathcal{L}}{\partial q_k} = -\sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k} = Q_k$$

FORCE OF CONSTRAINT

$$\mathcal{L} = \frac{1}{2}m(\dot x^2 + \dot z^2) - mgz$$

$$\mathcal{L} = \frac{1}{2}m\dot r^2 - \frac{1}{2}m r^2 \dot\theta^2 - mgr\sin\theta$$



CONSTRAINT $r - a = 0$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot r} - \frac{\partial \mathcal{L}}{\partial r} = -\lambda \frac{\partial(a-r)}{\partial r}$$

SAME FOR θ $= -\lambda \frac{\partial(a-r)}{\partial \theta}$

CONSTRAINT FORCES MUST DO ZERO WORK

CANONICAL / CONJUGATE MOMENTA

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot q_j}$$

CONSERVATION OF MOMENTA CONJUGATE TO CYCLIC PARAMETERS

ENERGY FN

$$h(q_i, \dot{q}_i, t) = \sum_j \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L}$$

$$\frac{dh}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$$ + INDEPENDENCE OF $\mathcal{L}$

$\Rightarrow h$ CONSERVED

$h = E$ IF T IS QUADRATIC IN $\dot{q}_i$

AND $V \neq V(\dot{q}_i)$

$$\Rightarrow \quad h = T + V = E$$

CENTRAL FORCES

2 OBJECTS + CENTRAL FORCE

$$\Rightarrow \quad \mathcal{L} = \tfrac{1}{2} M \dot{\vec{R}}^2 + \tfrac{1}{2}\mu \dot{\vec{r}}^2 - U(\vec{r}, \dot{\vec{r}})$$

WE LOOKED AT $V = V(r)$

ROTATIONALLY INVARIANT

$\vec{L} = \vec{r} \times \vec{p}$ CONSERVED

$$\mathcal{L} = \tfrac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$l_z = p_\theta = m r^2 \dot{\theta}$$

$$m\ddot{r} - \frac{l_z^2}{m r^3} = -\frac{\partial V}{\partial r}$$

$$V' = V + \frac{1}{2}\frac{l_z^2}{\mu r^2}$$ CENTRIPETAL BARRIER

1) $E > 0$ UNBOUND $V = -\frac{k}{r}$ HYPERBOLA

2) $E = 0$ UNBOUND " PARABOLA

3) $E < 0$ BOUND (r_1, r_2) " ELLIPSE

4) $E = E_{MIN} < 0$ BOUND r_{MIN} CIRCLE

$\hookrightarrow$ ECCENTRICITY, r_{MAX}, r_{MIN}, TURNING POINTS $\dot{r} = 0$

CLOSED ORBITS (ONLY FOR $V = +kr^2$, $V = -k/r$)

VIRIAL THEOREM $V = ar^{n+1}$

HO $\bar{T} = \frac{1}{2}\bar{V}$ GRAV $\bar{T} = -\frac{1}{2}\bar{V}$

OSCILLATIONS

SMALL OSC AROUND STABILITY

$$q_i = q_{0_i} + \eta_i$$

$$V(q_i) = V(q_{0_i}) + \cancel{\left.\frac{\partial V}{\partial q_i}\right|_0 \eta_i}^{\,0} + \frac{1}{2}\left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0 \eta_i \eta_j + \ldots$$

CONSTANT

$$= \frac{1}{2} V_{ij} \eta_i \eta_j$$ ← SYMMETRIC

$$T = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j \qquad T_{ij} = m_{ij}\big|_0$$

$$\Rightarrow T_{ij} \ddot{\eta}_j + V_{ij} \eta_j = 0$$

MOST PROBLEMS, WILL ~~BE~~ HAVE DIAGONAL T

$$T = \frac{1}{2} (\dot{\vec{\eta}})^T \overleftrightarrow{\mathbb{T}} \dot{\vec{\eta}}$$

$$V = \frac{1}{2} (\vec{\eta})^T \overleftrightarrow{\mathbb{V}} \vec{\eta}$$

$$\overleftrightarrow{\mathbb{T}} \ddot{\vec{\eta}} + \overleftrightarrow{\mathbb{V}} \vec{\eta} = \vec{0}$$

SOLUTION: OSCILLATORY ANSATZ

$$\eta_i = C a_i e^{-i\omega t}$$

$$-\omega^2 \overleftrightarrow{\mathbb{T}} \vec{a} + \overleftrightarrow{\mathbb{V}} \vec{a} = \vec{0}$$

$$\det\left(\overleftrightarrow{\mathbb{V}} - \lambda \overleftrightarrow{\mathbb{T}}\right) = 0 \qquad \lambda = \omega^2$$

$\exists$ EIGENVECTOR $\vec{a}_k$ FOR EACH λ_k

NORMAL MODES $\quad C\vec{a}_k \, e^{i\sqrt{\lambda_k}\, t}$

CAN ALSO DO ALGEBRA WITH

$$\overleftrightarrow{A} = \begin{pmatrix} (a_1 \cdots a_n)_1 \\ (a_1 \cdots a_n)_2 \\ \vdots \\ (a_1 \cdots a_n)_n \end{pmatrix}$$

SEE WEEKS NOTES

LINEAR TRIATOMIC MOLECULE
3D " "

ROTATIONS - RIGID BODY MOTION

KINEMATICS 6 DOF XYZ, ROLL, PITCH, YAW

ORTHOGANAL TRANSFORMS

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ACTIVE (ROTATE OBJECT)
PASSIVE (ROTATE COORDS)

FORMAL PROPERTIES

$C = AB$ $\quad (AB)C = A(BC)$

$AB \neq BA$

$A^{-1} = A^T$

EULER ANGLES ROT AROUND: z, THEN x'
THEN z'

FINITE AND INFINITESIMAL ROTATIONS

$\hat{n} \quad \phi \qquad \dot{\phi} = \omega(t) \qquad \vec{\omega} = \hat{n}\dot{\phi}$

$$\left.\frac{d\vec{V}}{dt}\right|_{space} = \left.\frac{d\vec{V}}{dt}\right|_{body} + \vec{\omega} \times \vec{V}$$

ROTATIONS

$$(d\vec{G})_{SPACE} = (d\vec{G})_{BODY} + (d\vec{G})_{ROT}$$

$$\vec{F}_{EFF} = \vec{F} - 2m(\vec{\omega} \times \vec{V}_r) - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

CORIOLIS CENTRIFUGAL

PARITY DISCUSSION

VECTOR $\vec{V} \to -\vec{V}$

AXIAL VECTORS $\vec{V}_a \to \vec{V}_a$

SCALARS $s \to s$

PSEUDO SCALARS $ps \to -ps$

$$\mathbb{I}_{jk} = \int \rho(\vec{r})\left(r^2 \delta_{jk} - x_j x_k\right) d^3r$$

$$\vec{L} = \overleftrightarrow{\mathbb{I}}\,\vec{\omega}$$

$\vec{L}$ NOT ALWAYS PARALLEL TO $\vec{\omega}$

$$I = \hat{n}^T \overleftrightarrow{\mathbb{I}}\,\hat{n} = \sum_i m_i\left(r_i^2 - (\vec{r}_i \cdot \hat{n})^2\right)$$

$$= \sum_i m_i r_{\perp i}^2$$

$$T = \frac{1}{2} I \omega^2$$

$\exists$ ORIGIN AND AXIS $\ni$ $\overleftrightarrow{\mathbb{I}}$ IS DIAGONAL

ORIGIN = CM

AXIS WILL BE ALONG A PRINCIPAL AXIS

PRINCIPAL AXES = EIGENVECTORS OF $\mathbb{I}$

PARALLEL AXIS THM

$$I_a = I_b + MR_\perp^2$$

EULER'S EQ OF MOTION (PRINCIPAL AXES)

$$\frac{d\vec{L}}{dt} = \vec{N} = \overleftrightarrow{I}\frac{d\vec{\omega}}{dt} + \vec{\omega} \times \left(\overleftrightarrow{I}\vec{\omega}\right)$$

TORQUE-FREE MOTION

TOPS

HAMILTONIANS

$$\mathcal{L}(q_i, \dot{q}_i, t) \rightarrow H(q_i, p_i, t)$$

$$H = \dot{q}_i p_i - \mathcal{L}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\mathcal{L} = T - V$$

CANONICAL TRANSFORMS

$$\{q_i, p_i\} \rightarrow \{Q_j, P_j\}$$

NEED $K(Q_j, P_j) \ni$

$$\frac{\partial K}{\partial Q_i} = -\dot{P}_i \qquad \frac{\partial K}{\partial P_i} = \dot{Q}_i$$

CHECKING IF TRANSFORM IS CANONICAL

a) GENERATING FN

b) SYMPLECTIC APPROACH

$$\left.\frac{\partial Q_i}{\partial q_j}\right)_{qp} = \left.\frac{\partial p_j}{\partial P_i}\right)_{QP} \qquad \left.\frac{\partial Q_i}{\partial p_j}\right)_{qp} = \left.-\frac{\partial q_j}{\partial P_i}\right)_{QP}$$

(COMPACT MATRIX NOTATION)

ANY INFINITESIMAL CANONICAL TRANSFORM IS SYMPLECTIC

c) POISSON BRACKETS

$$[u,v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

$[q_i, p_j] = \delta_{ij}$ INVARIANT UNDER CT

IF $[Q_i, P_j] = \delta_{ij} \Rightarrow$ CT WAS CANONICAL

PHASE SPACE AND PS INVARIANCE

CONTINUUM MECHANICS

$$\eta_i \rightarrow \eta(x)$$

X IS AN INDEX
NOT A COORDINATE

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}_i} - \frac{\partial \mathcal{L}}{\partial \eta_i} = 0$$

Now
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}} + \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial \eta'} - \frac{\partial \mathcal{L}}{\partial \eta} = 0$$

WAVES ON A STRING
" " " MEMBRANE

NORMAL MODES

RELATIVITY