Dr. Weinstein's Class Notes - Week 2

Transcribe by Ryan Guenthner

January 2021

1 Equations of Motion in Polar Coordinates

1.1 General Theory and Derivations

To begin, we recall the following definitions for generalized force

$$Q_j = \sum_i \mathbf{F} \cdot \frac{\partial \mathbf{r_i}}{\partial q_j} \tag{1}$$

and for the equations of motion (EOM)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_i} = Q_i \tag{2}$$

where T is the kinetic energy of the system. We can also make the substitution $T \rightarrow \mathcal{L}$ in the above equation, where $\mathcal{L}$ is the Lagrangian,

$$\mathcal{L} = T - V \tag{3}$$

in which case we omit the term Q_i (unless we have a really irritating problem with both potentials and forces). We will now set about deriving a general expression for T in polar coordinates. In Cartesian coordinates, the kinetic energy can be written

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \tag{4}$$

which we can convert to polar coordinates using the standard definitions

$$x = r\cos\theta \quad , \quad y = r\sin\theta \tag{5}$$

Assuming r and θ are time-dependent, we can use the chain rule to write

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \quad , \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{6}$$

Next, we require the square of each term in (6)

$$\dot{x}^2 = \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\dot{\theta}^2\sin^2\theta \tag{7}$$

$$\dot{y}^2 = \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\dot{\theta}^2\cos^2\theta \tag{8}$$

Finally, substituting (7) and (8) into (4) and simplifying, we find our expression for the kinetic energy in polar coordinates.

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{9}$$

Before deriving the equations of motion which correspond to (9), it is informative to calculate the components of the generalized force using (1). For the radial component, using (5) we find

$$
\begin{aligned}
Q_r &= F_x \frac{\partial x}{\partial r} + F_y \frac{\partial y}{\partial r} \\
&= F_x \cos\theta + F_y \sin\theta \\
&= \mathbf{F} \cdot \hat{\mathbf{r}}
\end{aligned} \tag{10}
$$

which is simply the force in the radial direction. For the θ component we find

$$
\begin{aligned}
Q_\theta &= F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \\
&= -F_x r \sin\theta + F_y r \cos\theta \\
&= -y F_x + x F_y \\
&= |\mathbf{r} \times \mathbf{F}| \\
&= \mathbf{N}
\end{aligned} \tag{11}
$$

which is the torque. We now proceed to write the equations of motion for each component using Lagrange's equation. In the radial direction, we find

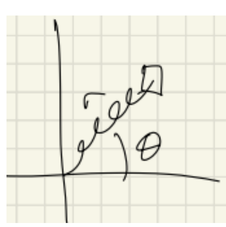
$$
\begin{aligned}
Q_r &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r} \\
&= m \frac{\mathrm{d}}{\mathrm{d}t} \dot{r} - m r \dot{\theta}^2 \\
&= m \ddot{r} - m r \dot{\theta}^2
\end{aligned} \tag{12}
$$

where the second term on the right is the familiar centripetal force. For the θ component we find

$$
\begin{aligned}
Q_\theta &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} \\
&= m \frac{\mathrm{d}}{\mathrm{d}t} r^2 \dot{\theta} \\
&= m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta}
\end{aligned} \tag{13}
$$

where the second term on the right is again a familiar one - the Coriolis force.

1.2 Example: Rotating Mass on a Spring



We now consider the problem of a spring of length l attached to the origin at one end, and attached to a block of mass m at the other. Assume the spring is able to rotate freely in the $\hat{\theta}$ direction. The kinetic energy term for this problem is identical to the one derived above, we need only write the potential term, which for a spring is given by

$$V = \frac{1}{2}k(r-l)^2 \tag{14}$$

where k is the traditional spring constant. Using (9) and (14) we write our Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}k(r-l)^2 \tag{15}$$

Since we know the specific potential for this system, there is no need to construct a generalized force. Additionally, since the potential does not involve θ, the angular momentum will be constant, and there will be no net torque on the system. The equation of motion reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = 0 \tag{16}$$

And the radial EOM is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + k(r-l) = 0 \tag{17}$$

Stationary Solution

In order to obtain the stationary solution we set $\ddot{r} = \ddot{\theta} = 0$, which implies both $\dot{r}$ and $\dot{\theta}$ are constants. We define $\dot{\theta} = \omega_0$, and the radial EOM reads

$$\begin{aligned} mr\omega_0^2 &= k(r-l) \\ \omega_0^2 &= \frac{k(r-l)}{mr} \end{aligned} \tag{18}$$

for $r > l$ (since there is no stationary solution corresponding to the spring being compressed).

Small Oscillations

For small oscillations, we assumed the system experiences a small perturbation which we can write as

$$r(t) = r_0 + \delta r \quad , \qquad \delta r << r_0 \tag{19}$$

$$\dot{\theta}(t) = \omega_0 + \delta\dot{\theta} \quad , \qquad \delta\dot{\theta} << \omega_0 \tag{20}$$

Using these, the radial EOM becomes

$$m\delta\ddot{r} - m(r_0 + \delta r)(\omega_0^2 + 2\omega_o\delta\dot{\theta} + \delta\dot{\theta}^2) + k(r_0 + \delta r - l) = 0 \tag{21}$$

We are interested in small oscillations around the stationary state, and we will use the following results from our stationary state analysis

$$mr_0\omega_0^2 = k(r_0 - l) \tag{22}$$

and

$$mr^2\dot{\theta} = mr_0^2\omega_0 \tag{23}$$

substituting (19) and (20) into (23) yields

$$m(r_0 + \delta r)^2(\omega_0 + \delta\dot{\theta}) = mr_0^2\omega_0 \tag{24}$$

Since we are dealing with small oscillations, we omit all terms that are higher than first order in δr and $\delta\dot{\theta}$. (24) is then rewritten and solved for $\delta\dot{\theta}$

$$\begin{aligned}
mr_0^2\omega_0 &= mr_0^2\omega_0 + 2mr_0\delta r\omega_0 + mr_0^2\delta\dot{\theta} \\
mr_0^2\delta\dot{\theta} &= -2mr_0\delta r\omega_0 \\
\delta\dot{\theta} &= -\frac{2\omega_0}{r_0}\delta r
\end{aligned} \tag{25}$$

We now return to (21), which when neglecting higher order terms is written

$$m\delta\ddot{r} = mr_0\omega_0^2 - k(r_0 - l) + 2mr_0\delta\dot{\theta}\omega_0 + m\delta r\omega_0^2 - k\delta r \tag{26}$$

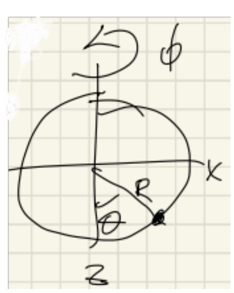
Finally, we can use (22) and (24) to write

$$\begin{aligned}
m\delta\ddot{r} &= -2mr_0\omega_0\left(\frac{2\omega_0}{r_0}\delta r\right) + m\delta r\omega_0^2 - k\delta r \\
&= \delta r(m\omega_0^2 - k - 4m\omega_0^2) \\
&= -\delta r(3\omega_0^2 + k)
\end{aligned} \tag{27}$$

Where the form of this differential equation should be familiar to the reader, and the frequency of small oscillations is easily seen to be

$$\omega_r = \sqrt{3\omega_0^2 + {}^k\!/\!_m} \tag{28}$$

1.3 Example: Bead on a Hoop



Using spherical coordinates, we define the azimuthal frequency as

$$\Omega = \dot{\phi} \tag{29}$$

The kinetic energy of the hoop is given by

$$T = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m(\Omega R \sin\theta)^2 \tag{30}$$

where the angle θ has been defined such that the axial portion of the kinetic energy provides its maximum contribution at $\theta = \pi/2$. The potential energy is gravitational, and is written

$$V = -mgR\cos\theta \tag{31}$$

where the negative sign assures we have a positive value for the energy at $\theta = \pi$. Our Lagrangian is

$$\mathcal{L} = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m(\Omega R \sin\theta)^2 + mgR\cos\theta \tag{32}$$

which depends only a single generalized coordinate θ. The EOM are then given by

$$mR^2\ddot{\theta} = m\Omega^2 R^2 \sin\theta\cos\theta - mgR\sin\theta \tag{33}$$

which simplifies to

$$\ddot{\theta} = \sin\theta\Big(\Omega^2\cos\theta - \frac{g}{R}\Big) \tag{34}$$

Limiting Case 1: Small angles, no rotation

In this case, $\Omega = 0$ and $\sin\theta \approx \theta$. The EOM then reduces to

$$\ddot{\theta} = -\frac{g}{R}\theta \tag{35}$$

Again, the form of this equation should be familiar to the reader, and the motion can be written

$$\theta = A\cos\Big(\sqrt{g/R} + \delta\Big) \tag{36}$$

where A is an arbitrary constant, δ is a phase shift, and the frequency is given by $\omega = \sqrt{g/R}$.

Limiting Case 2: Stationary solution for non-zero rotation

For the stationary solution ($\ddot{\theta} = 0$) corresponding to a non-zero rotation ($\Omega \neq 0$), the EOM is written

$$0 = \sin\theta\Big(\Omega^2\cos\theta - \frac{g}{R}\Big) \tag{37}$$

There are two solutions which satisfy this condition. The first is given by

$$g = \Omega^2 R \cos\theta \qquad \text{or} \qquad \cos\theta = \frac{g}{\Omega^2 R} \tag{38}$$

which is valid only if $g/\Omega^2 R \leq 1$, which is a limit imposed by the range of $\cos\theta$. The other solution occurs is $\sin\theta = 0$, which occurs at either $\theta = 0, \theta = \pi$. At $\theta = \pi$ the bead is at the top of the hoop, and the equilibrium is unstable. At $\theta = 0$ the nature of the equilibrium is determine by the ratio of g/R.

Small Oscillations and the Stationary Solution

We define the perturbation

$$\theta = \theta_0 + \delta\theta \tag{39}$$

our equation of motion is then

$$R\delta\ddot{\theta} = \sin(\theta_0 + \delta\theta)\big(\Omega^2 R \cos(\theta_0 + \delta\theta) - g\big) \tag{40}$$

To analyze this equation, we will need the following trig identities

$$\sin(\theta_0 + \delta\theta) = \sin\theta_0 \cos\delta\theta + \sin\delta\theta\cos\theta_0 \tag{41}$$
$$\cos(\theta_0 + \delta\theta) = \cos\theta_0 \cos\delta\theta - \sin\theta_0 \sin\delta\theta \tag{42}$$

and for small angles we will use

$$\cos\delta\theta \approx 1$$
$$\sin\delta\theta \approx \delta\theta$$

Our EOM then becomes

$$R\delta\ddot{\theta} = (\sin\theta_0 + \cos\theta_0\delta\theta)(\Omega^2 R(\cos\theta_0 - \sin\theta_0\delta\theta) - g) \tag{43}$$

If we only work to first order in $\delta\theta$, this reduces to

$$R\delta\ddot{\theta} = \sin\theta_0(\Omega^2 R\cos\theta_0 - g) + \delta\theta(\Omega^2 R\cos^2\theta_0 - g\cos\theta_0 - \Omega^2 R\sin^2\theta_0) \tag{44}$$

Using the stationary solutions (38) this further reduces to

$$\delta\ddot{\theta} = -\Omega^2 \sin^2\theta_0 \delta\theta \tag{45}$$

and finally solving the differential equation yields the frequency of small oscillations

$$\text{freq} = \Omega\sin\theta_0 = \sqrt{\Omega^2 - (g^2/\Omega^2 R^2)} \tag{46}$$

Forces of Constraint

We already know that since the bead is confined to the hoop at radius R, which means that $\delta r = 0$. However, we are now interested in find the constraint force F_r such that $\delta r = 0$. We begin with the standard definition

$$R = R + \delta r \tag{47}$$

Working only to first order in δr the kinetic energy can be written

$$T = \frac{m}{2}\Big(\delta\dot{r}^2 + (R^2 + 2R\delta r)(\dot{\theta}^2 + \Omega^2 \sin^2\theta)\Big) \tag{48}$$

and the potential energy becomes

$$V = mg(R + \delta r)\cos\theta - F_r \delta r \tag{49}$$

where the last term represents the work done by the force of constraint. Our EOM is obtained from

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \delta\dot{r}} - \frac{\partial \mathcal{L}}{\partial \delta r} = 0 \tag{50}$$

which yields

$$m\delta\ddot{r} = mR(\dot{\theta}^2 + \Omega^2\sin^2\theta) - mg\cos\theta - F_r \tag{51}$$

However, δr is constrained which means $\delta\ddot{r} = 0$. Finally we can use this to solve the above equation for the constraint force, which we find to be

$$F_r = mR(\dot{\theta}^2 + \Omega^2\sin^2\theta) - mg\cos\theta \tag{52}$$

The result is the difference between the centripetal force and the force of gravity. We now turn our attention to the constraint force F_θ. We define

$$\Omega = \Omega_0 + \delta\dot{\phi} \tag{53}$$

Our kinetic and potential energies are then written

$$T = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\sin^2\theta(\Omega_0 + \delta\dot{\phi})^2) \tag{54}$$

$$V = -mgR\cos\theta + N\delta\phi \tag{55}$$

we are already aware the constraint force is proportional to the torque, which allows us to write the potential as

$$V = -mgR\cos\theta + R\sin\theta F_\phi \delta\phi \tag{56}$$

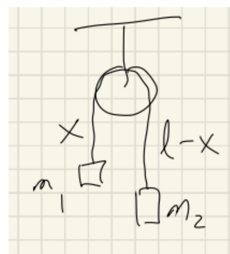
Working to first order $\delta\dot{\phi}$, our EOM becomes

$$2mR^2\cos\theta\sin\theta(\Omega_0 + \delta\phi)\dot{\theta} + mR^2\sin^2\theta\delta\ddot{\phi} - R\sin\theta F_\phi = 0 \tag{57}$$

Working with constraints allows us to set $\delta\phi = \delta\ddot{\phi} = 0$, after which we can finally solve for F_ϕ

$$F_\phi = -2mR\Omega_0 \cos\theta\dot{\theta} \tag{58}$$

1.4 Example: Atwood's Machine



We now turn our attention to one of the earliest problems encountered in introductory physics. Using the same procedures we have developed exhaustively above, we demonstrate how we can use the Lagrangian to very simply obtain the EOM.

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 \tag{59}$$

$$V = -m_1 g x - m_2 g(l - x) \tag{60}$$

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 g x + m_2 g(l - x) \tag{61}$$

$$(m_1 + m_2)\ddot{x} - (m_1 g - m_2 g) = 0 \tag{62}$$

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g \tag{63}$$

2 Velocity Dependent Potentials

If there exists a function $U(q_j, \dot{q}_j)$ such that the generalized forces can be obtain using

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \tag{64}$$

then the EOM can still obtained from a Lagrangian of the form

$$\mathcal{L} = T - U \tag{65}$$

Such potentials are called "velocity-dependent potentials" or "generalized potentials".

2.1 Lorentz Force

One common example of a force which can be written at a velocity dependent potential is the Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{66}$$

Where the $\mathbf{E}$ and $\mathbf{B}$ fields can be expressed in terms of the scalar potential ϕ and the vector potential $\mathbf{A}$.

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{67}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{68}$$

As it turns out, the potential which gives rise to the Lorentz force is given by

$$U = q(\phi - \mathbf{A} \cdot \mathbf{v}) \tag{69}$$

and the corresponding Lagrangian is written

$$\mathcal{L} = \frac{1}{2}mv^2 - q\phi + q\mathbf{A} \cdot \mathbf{v} \tag{70}$$

In order to demonstrate this does indeed give rise to the Lorentz force, we need only prove it is true for one component, which we arbitrarily choose to be the x-component. As such, we can write the quantity $\mathbf{A} \cdot \mathbf{v}$ as $A_x\dot{x}$ which gives us

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + qA_x \tag{71}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} + q\frac{\mathrm{d}A_x}{\mathrm{d}t} \tag{72}$$

Using the chain rule, we can rewrite the last term as

$$\frac{\mathrm{d}A_x}{\mathrm{d}t} = \frac{\partial A_x}{\partial t} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z} \tag{73}$$

which can be used to rewrite (72) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} + q\left(\frac{\partial A_x}{\partial t} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}\right) \tag{74}$$

Next, recalling that both ϕ and $\mathbf{A}$ are functions of position as well as time, we find

$$\frac{\partial \mathcal{L}}{\partial x} = -q\frac{\partial \phi}{\partial x} + q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) \tag{75}$$

We at last obtain our EOM

$$m\ddot{x} = q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right) - q\frac{\partial \phi}{\partial x} - q\left(\frac{\partial A_x}{\partial t} + v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}\right) \tag{76}$$

Before proceeding, we observe that

$$\begin{aligned}(\mathbf{v}\times\mathbf{B})_x &= \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})_x \\ &= v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) \end{aligned}\tag{77}$$

Finally, we can use this to write the x-component of the EOM in the familiar form

$$m\ddot{x} = q\left[-\frac{\partial \phi}{\partial x} + v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right] \tag{78}$$

$$= q[E_x + (\mathbf{v}\times\mathbf{B})_x] \tag{79}$$

Showing that this holds true for one component in Cartesian coordinates is equivalent to proving the statement.

2.2 Rayleigh Dissipation Function

Another velocity dependent force commonly come across in physics is drag force. Consider a velocity-dependent force in the x-direction, given by

$$F_x = -k_x v_x \tag{80}$$

The equations of motion for such a force can be derived from Rayleigh's dissipation function

$$\mathcal{F} = \frac{1}{2}\sum_i \left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right) \tag{81}$$

where the resultant force is obtained via

$$\mathbf{F} = -\boldsymbol{\nabla}_v \mathcal{F} \tag{82}$$

Physically this can be interpreted as the work done against drag by the system

$$\mathrm{d}W = -\mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\mathbf{F}\cdot\mathbf{v}\,\mathrm{d}t = \left(k_x v_x^2 + k_y v_y^2 + k_z v_z^2\right)\mathrm{d}t \tag{83}$$

which implies that $2\mathcal{F}$ is the rate of energy dissipation due to friction. We can also find the component of the generalized force due to drag

$$Q_j = \sum_i \mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \boldsymbol{\nabla}_v\mathcal{F}\cdot\frac{\partial \mathbf{r}_i}{\partial q_j} \tag{84}$$

recalling an identity proved in an earlier lecture

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \tag{85}$$

which we can use to write

$$\begin{aligned} Q_j &= -\sum_i \boldsymbol{\nabla}_v \mathcal{F} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \\ &= -\sum_i \frac{\partial \mathcal{F}}{\partial v} \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \\ &= -\frac{\partial \mathcal{F}}{\partial \dot{q}_j} \end{aligned} \tag{86}$$

The EOM is then written

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0 \tag{87}$$

Where the reader can see that in order obtain the EOM, two scalar functions, $\mathcal{L}$ and $\mathcal{F}$ must be specified. A common example of a force which can be derived from a velocity-dependent potential is Stokes law for a sphere of radius a moving at a speed v through a fluid of viscosity η. The drag force is given by

$$\mathbf{F} = -6\pi\eta a \mathbf{v} \tag{88}$$

3 Hamilton's Principle

Assuming a force is derivable from a scalar potential of the form

$$V(\{q_i\}, \{\dot{q}_i\}, t) \tag{89}$$

Then the action I, given by

$$I = \int_{t_1}^{t_2} \mathcal{L}\,\mathrm{d}t \tag{90}$$

is stationary with respect to the path. This implies the action remains unchanged if the path undergoes small variations. Consider an initial path $q_i(t)$, which then undergoes a slight perturbation $\alpha\eta_i(t)$. We write the new path as

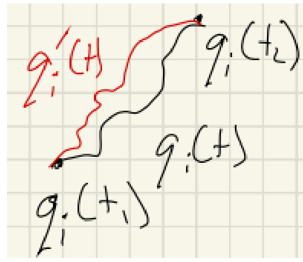
$$q_i'(t) = q_i(t) + \alpha\eta_i(t) \tag{91}$$

The action with respect to the new path is then written

$$I = \int_{t_1}^{t_2} \mathcal{L}(\{q_i(t) + \alpha\eta_i(t)\}, \{\dot{q}_i + \alpha\dot{\eta}_i(t)\}, t)\,\mathrm{d}t \tag{92}$$

We also assume the perturbation does not affect the endpoints of the path, that is

$$\eta_i(t_1) = \eta_i(t_2) = 0 \tag{93}$$



In order for the action to remain stationary, its derivative with respect to the perturbation must be equal to zero, that is

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{t_1}^{t_2} \mathcal{L}(\{q_i(t) + \alpha\eta_i(t)\}, \{\dot{q}_i + \alpha\dot{\eta}_i(t)\}, t)\,\mathrm{d}t = 0 \tag{94}$$

Using the chain rule, the integrand can be expressed as

$$\begin{aligned} &\sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\frac{\partial}{\partial \alpha}(q_i + \alpha\eta_i) + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial}{\partial \alpha}(\dot{q}_i + \alpha\dot{\eta}_i)\right) \\ &= \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\eta_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{\eta}_i\right) \end{aligned} \tag{95}$$

again we can use the chain rule to rewrite the second term. We note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\eta_i\right) = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{\eta}_i + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\eta_i \tag{96}$$

which can be used to rewrite the second term in (95). We can know write the full integral as follows

$$\sum_i \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\eta_i\right)\mathrm{d}t + \sum_i \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right)\eta_i\,\mathrm{d}t = 0 \tag{97}$$

The first term is merely the function evaluated at the endpoints. However, we have already defined the behavior of η at the endpoints (93), and we know that terms must necessarily be zero. We are left with

$$\sum_i \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right)\eta_i\,\mathrm{d}t = 0 \tag{98}$$

which requires the (familiar) term in parenthesis to be zero in order to be satisfied. Thus we have derived the EOM from Hamilton's principle rather than D'Alembert's.