Classical Mechanics Lecture Notes

Tommaso Rainaldi
train005@odu.edu

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Examples of Calculus of variations

Given any functional $J[y]$ in the form of

$$J[y] = \int_{x_1}^{x_2} dx f(x, y, y'), \tag{1}$$

with the unknown function y satisfying $y(x_1) = y_1$ and $y(x_2) = y_2$, we know to be minimized if the following condition is satisfied, namely

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0, \tag{2}$$

where $y' = dy/dx$. Let's study a couple of examples.

1. What is the function $y(x)$ that minimizes the distance between two given points?

   Let us work in two dimensions for simplicity, thus the infinitesimal path element simply reads

   $$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + (\frac{dy}{dx})^2} = dx\sqrt{1 + y'^2} \equiv f(x, y, y')dx \tag{3}$$

   The distance $D[y]$ between the two points is simply

   $$D[y] = \int_1^2 ds = \int_{x_1}^{x_2} dx\sqrt{1 + y'^2}. \tag{4}$$

   Solving the associated Lagrange's equation yields

   $$\frac{d}{dx}\Big(\frac{y'}{\sqrt{1 + y'^2}}\Big) = 0, \tag{5}$$

   which can only be true if y' is constant, namely

   $$y(x) = ax + b, \qquad a, b \in \mathbf{R}. \tag{6}$$

   This result should come to no surprise since we know that the shortest path between two points is indeed a straight line, however, this fact has been elegantly verified and confirmed using this new technique as well.

2. What is the curve that minimizes the time a sliding object takes while moving from its initial point to its final point both situated at the same height?

   From simple physical considerations we have that it must be

$$\frac{1}{2}mv^2 = mgy \quad \rightarrow \quad v = \sqrt{2gy}, \tag{7}$$

   where we are taking the positive y-axis to point downwards.
   This time the functional we want to study is the total time $T[y]$ between the initial and final points, or equivalently

$$T[y] = \int_{t_1}^{t_2} dt = \int_1^2 \frac{ds}{v} = \int_{x_1}^{x_2} dx \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}, \tag{8}$$

   therefore we have now

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}, \tag{9}$$

   which must satisfy Lagrange's equation and leads to the following parametric solution, namely

$$x = a(\phi - \sin\phi), \quad y = a(1 - \cos\phi). \tag{10}$$

   This is actually a very famous problem in the history of mathematics known as the brachistochrone problem (literally "shortest time"), whose solution represents a cycloid.

Extending Hamilton's Principle to system with constraints

Up until now we have solved both physical and mathematical problems by exploiting the results obtained from Hamilton's Principle, however, we have always assumed the infinitesimal variations of the generalized coordinates to be independent with one another, namely the $\{\delta q_i\}$.
What happens if that is no more valid? In other words, how are we to proceed if the problem/system has to satisfy some constraints?
We can show that Hamilton's Principle can still be used to solve systems with holonomic constraints and also certain types of non-holonomic ones.
Let us start by expressing the constraint equations as follows

$$f_j(\{q_i\}) = 0, \quad j = 1, .., k; \; i = 1, ..., n; \quad n > k, \tag{11}$$

namely we have k constraint equations and n generalized coordinates. Note also that we haven't been as general as we could be, because we have assumed the constraints to be time independent, however this doesn't necessarily hold in general. However, this is enough for our purpose and, from the above equation we can thus safely state that it must be

$$0 = \delta f_j = \sum_i^n \frac{\partial f_j}{\partial q_i} \delta q_i, \tag{12}$$

since we are working with a vanishing quantity. Moreover, we can multiply each term by an unknown parameter λ_j (a.k.a. the Lagrange multipliers), i.e.:

$$0 = \lambda_j \delta f_j = \lambda_j \sum_i^n \frac{\partial f_j}{\partial q_i} \delta q_i \quad \rightarrow \quad \sum_j^k \lambda_j \sum_i^n \frac{\partial f_j}{\partial q_i} \delta q_i = 0 \tag{13}$$

Therefore, we can integrate over time and still get zero, namely

$$0 = \int_{t_1}^{t_2} dt \Big(\sum_j^k \lambda_j \sum_i^n \frac{\partial f_j}{\partial q_i} \delta q_i \Big) \tag{14}$$

This simple but clever trick allows us to add such expression to the usual Lagrange's equations because, formally, we are simply adding zero. However, this also makes it so the $\{\delta q_i\}$ dependence due to the constraints is shifted to the new added Lagrange multipliers, thus making them effectively independent quantities. With this said we finally have

$$\delta S = \int_{t_1}^{t_2} dt \sum_i^n \Big(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} + \sum_j^k \lambda_j \frac{\partial f_j}{\partial q_i} \Big) \delta q_i = 0, \tag{15}$$

which translates into the following modified equations of motion, i.e.:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} + \sum_j^k \lambda_j \frac{\partial f_j}{\partial q_i} = 0. \tag{16}$$

Note that we have now n generalized coordinates and k Lagrange multipliers, thus making it $n+k$ variables but only n are independent.
We can also interpret the constraints using the effective forces method, namely we write

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i + Q_i^{constraint}, \tag{17}$$

where

$$\begin{aligned} Q_i &= -\frac{\partial V}{\partial q_i}, \qquad \Big(\frac{\partial V}{\partial \dot{q}_i} = 0 \Big) \\ Q_i^{constraint} &= \sum_j^k \lambda_j \frac{\partial f_j}{\partial q_i}, \end{aligned} \tag{18}$$

or equivalently

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i^{constraint}. \tag{19}$$

Let us look at some applications of holonomic and non-holonomic constraints.

Holonomic constraint:

1. Let us consider a sliding mass down an inclined plane. Its Lagrangian reads

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy, \tag{20}$$

and the constraint equation reads instead

$$f(x, y) = y - x \tan\theta = 0, \tag{21}$$

namely the mass doesn't jump off the plane. Since we have only one constraint, we only need one Lagrange multiplier λ and make the replacement

$$\mathcal{L} \to \mathcal{L} + \lambda(y - \tan\theta x). \tag{22}$$

From the Euler-Lagrange equations we get

$$\begin{cases} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} = \frac{\partial \mathcal{L}}{\partial x} = -\lambda \tan\theta \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\ddot{y} = \frac{\partial \mathcal{L}}{\partial y} = -mg + \lambda \end{cases} \tag{23}$$

which, along with constraint equation (21) gives the following expression for λ:

$$\lambda = \frac{mg}{1+\tan^2\theta} = mg\cos^2\theta \tag{24}$$

and thus

$$\begin{cases} \ddot{x} = -g\sin\theta\cos\theta \\ \ddot{y} = -g\sin^2\theta. \end{cases} \tag{25}$$

This makes immediate sense if we go to the "tilted" corrdinate system where

$$x = x'\cos\theta. \quad y = y'\sin\theta, \tag{26}$$

because we have

$$\ddot{x}' = -g\sin\theta, \quad \ddot{y}' = -g\cos\theta. \tag{27}$$

Apart from this simple verification we want to study the constraint forces acting on the sliding mass, namely

$$\begin{aligned} F_x &= \lambda\frac{\partial f}{\partial x} = mg\cos^2\theta(-\tan\theta) = -mg\cos\theta\sin\theta = (m\ddot{x}) \\ F_y &= \lambda\frac{\partial f}{\partial y} = mg\cos^2\theta = (m\ddot{y} + mg) \neq (m\ddot{y}), \end{aligned} \tag{28}$$

where we immediately notice that the constraint force in the x-axis is exactly equal to its acceleration whereas this is not true for the y component. The reason for this is due to the fact that there is an actual force acting in the y direction, namely the gravitational field g, thus that is expected.

2. Mass m on top of a semi-hoop of radius a with gravity.
   The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgy, \tag{29}$$

and the constraint

$$f(x,z) = a - \sqrt{x^2+z^2} = 0. \tag{30}$$

With this kind of system we should work using spherical coordinates so that the two equations above become

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\cos\theta \tag{31}$$

and

$$f(x,z) = a - r = 0. \tag{32}$$

The equation for the radial coordinate reads

$$\begin{gathered} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = -\lambda\frac{\partial}{\partial r}(a-r) \\ \downarrow \\ m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda \end{gathered} \tag{33}$$

and the one for the angle reads instead

$$\begin{gathered}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = -\lambda \frac{\partial}{\partial \theta}(a-r) = 0 \\ \downarrow \\ 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} - mgr\sin\theta = 0.\end{gathered} \tag{34}$$

However, by simply differentiating the constraint equation with respect to time we find that it must be $\dot{r} = 0$ and thus the two above equation drastically simplify. In particular we have

$$\begin{gathered}\ddot{\theta} = \frac{g}{a}\sin\theta \\ ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0.\end{gathered} \tag{35}$$

The solution is found to be

$$\dot{\theta}^2 = 2\frac{g}{a}(1-\cos\theta), \tag{36}$$

with

$$\lambda = mg(3\cos\theta - 2). \tag{37}$$

The force of constraint must be positive but we clearly see that this is not always the case, in particular we have that when $3\cos\theta > 2$ the mass flies off the dome, namely when $\cos\theta = 2/3$. Therefore this problem has a holonomic constraint but only up to a certain point.

Semi-holonomic constraint:
Generally non holonomic constraints cannot be expressed using the variational principle, however, an exception is found when the constraint equation is linear in the generalized velocities, namely

$$f_\alpha = \sum_{k=1}^{n} a_{\alpha k}(\{q_i\}, t)\dot{q}_k + a_{\alpha 0}(\{q_i\}, t). \tag{38}$$

In this case we can treat the variation as in the previous cases, namely

$$\delta \int_{t_1}^{t_1} dt\Big(\mathcal{L} + \sum_{\alpha}^{m} \mu_\alpha f_\alpha\Big) = 0 \tag{39}$$

resulting in

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k = -\sum_{\alpha}^{m} \mu_\alpha \frac{\partial f_\alpha}{\partial \dot{q}_k}}. \tag{40}$$

Let us look at an example of such case.

We specialize in the motion of a hoop of radius R rolling down an inclined plane of angle α from height h. In particular, we are going to use the angle of rotation of the hoop θ and the distance it travels $s = \sqrt{x^2 + y^2}$, as the two generalized coordinates. Interestingly this problem can be formulated in a holonomic and non holonomic version, namely we have

$$\begin{gathered}f_{holonomic} = s - R\theta = 0 \\ f_{non-holonomic} = \dot{s} - R\dot{\theta} = 0\end{gathered} \tag{41}$$

The non-holonomic equation constrains the hoop not to slip over the surface of the inclined plane, namely the radial velocity of the hoop must equal its velocity along the inclined plane. The Lagrangian of the system reads

$$\mathcal{L} = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}I\dot{\theta}^2 - (h - s \cdot \sin\alpha)mg, \tag{42}$$

with $I = mR^2$.
From the equation of motion for s we get

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{s}} - \frac{\partial\mathcal{L}}{\partial s} &= -\lambda\frac{\partial f}{\partial\dot{s}} \\ &\downarrow \\ m\ddot{s} - mg\sin\alpha + \lambda &= 0, \end{aligned} \tag{43}$$

and for θ we instead obtain

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} &= -\lambda\frac{\partial f}{\partial\dot{\theta}} \\ &\downarrow \\ mR^2\ddot{\theta} - \lambda R &= 0. \end{aligned} \tag{44}$$

By differentiating the constraint equation once with respect to time we get that $R\ddot{\theta} = \ddot{s}$ and thus we finally obtain

$$\begin{cases} \ddot{s} = \frac{1}{2}g\sin\alpha \\ \ddot{\theta} = \frac{g}{2R}\sin\alpha \\ \lambda = \frac{1}{2}mg\sin\alpha, \end{cases} \tag{45}$$

where the last equation has to be interpreted as the frictional force acting on the rim of the hoop. Namely, the hoop doesn't slip because of that force.

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Noether's Theorem and conserved quantities

Given a Lagrangian formulation of the physical problem we want to study, we are now able to find its associated equations of motion. However, sometimes we may not be interested in solving them, be it because they are too complicated or for some other reason. The point is that, even if we do not solve the obtained equations, we can still obtain information about the system simply making use of "symmetry" considerations. What do we mean by this?
There is a very elegant and powerful theorem by the German mathematician Emmy Noether, who discovered that *any transformation that leaves the laws of nature unchanged has a corresponding conserved quantity*. Restating the above in a slightly more mathematical way would be: "Any transformation of the generalized coordinates that leaves the equations of motion invariant corresponds to a constant of motion".
Before we delve into the mathematics, let us state some powerful result that we can obtain by using such theorem, namely

$$\begin{aligned} \text{Time independence} &\iff \text{Energy conservation} \\ \text{Translation invariance} &\iff \text{Momentum conservation} \\ \text{Rotantion invariance} &\iff \text{Angular momentum conservation} \end{aligned} \tag{46}$$

In order to start the discussion about the conserved quantities we need to generalize the notion of momentum. Its common definition using Newtonian mechanics is simply the product of the mass and its velocity, namely $p = mv$. Obviously, we can find the same result using the Lagrangian formulation of, say, a system of N point masses under the influence of a position dependent only potential, i.e. $V = V(\{r_i\})$. The Lagrangian of such system reads

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - V(\mathbf{r}_i), \tag{47}$$

then it is

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \underbrace{\frac{\partial V}{\partial \dot{x}_i}}_{=0} = \frac{\partial}{\partial \dot{x}_i} \sum_{j=1}^{N} \frac{1}{2} m_j(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2) = m_i \dot{x}_i = p_i. \tag{48}$$

Therefore, it seems that taking the partial derivative of the Lagrangian with respect to the velocity gives the sought momentum. The generalization is thus straightforward. Instead of differentiating with respect to the Cartesian velocity, we simply define the generalized momentum, also known as *canonical* or *conjugate momentum* as follows

$$p_j \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_j}, \tag{49}$$

where the $\{\dot{q}_i\}$ are the derivatives with respect to time of the generalized coordinates. Immediately we notice that this new notion of momentum doesn't necessarily have the dimensions of a momentum. However, we immediately recover Newton's second Law in the case of a conservative potential, namely we have

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{d}{dt} p_i = \frac{\partial \mathcal{L}}{\partial q_i} = -\frac{\partial V}{\partial q_i} \quad \rightarrow \quad \dot{p}_i = F_i. \tag{50}$$

What if the potential is not only a function of the position but it also depends on the velocities? Surely the above is no longer valid, in the sense that even the generalized momentum associated with the Cartesian coordinates is no longer the linear momentum mass times velocity.
A clear example of this is given by the system of N particles under the influence of an electromagnetic field. In such case we have

$$\mathcal{L} = \sum_{i}^{N} \frac{1}{2} m_i \dot{r}_i^2 - \sum_{i}^{N} q_i \phi_i(\mathbf{r}_j) + \sum_{i}^{N} q_i \mathbf{A}(\mathbf{r}_j) \cdot \dot{\boldsymbol{r}}_i, \tag{51}$$

therefore the, say, x-component of the conjugate momentum of the i^{th} particle reads

$$p_{x,i} = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m_i \dot{x}_i + q_i A_x \neq m_i \dot{x}_i, \tag{52}$$

which clearly shows an additional term due to the vector potential. The linear momentum thus gets modified by the presence of the electromagnetic field and a redefinition is needed.

We haven't exhausted any possible scenarios. In fact, we could also have that our Lagrangian only contains the velocity $\dot{q}_i$ but not the corresponding coordinate q_i. In such case, the latter is said to be *cyclic* or *ingnorable* and the Euler-Lagrange equation reads

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \rightarrow \quad p_i = \text{constant}. \tag{53}$$

This immediately tells us that, if a coordinate is cyclic, its corresponding conjugate momentum is conserved.
Let's look at a trivial example:

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y) \tag{54}$$

The Lagrangian does not depend on the z coordinate therefore we have

$$\frac{\partial \mathcal{L}}{\partial z} = 0 \quad \rightarrow \quad p_z = m\dot{z} = \text{ conserved.} \tag{55}$$

Another slightly less trivial example is the 2D motion of a particle in a central potential, namely

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r). \tag{56}$$

In this case the Lagrangian does not depend on the angle ϕ and thus

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0 \quad \rightarrow \quad p_\phi = mr^2\dot{\phi} = L_z \text{ is conserved,} \tag{57}$$

namely, the $z-$component of the angular momentum is conserved whereas, the radial momentum

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \tag{58}$$

is not conserved and, in fact, must satisfy

$$\dot{p}_r = m\ddot{r} = \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 - \frac{\partial V}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}. \tag{59}$$

In the past we derived the conservation of momentum from Newton's third Law, however, this time we have no need for that. In fact, the results we have found are even more general and much stronger than that. Now we know that if the Lagrangian that describes the system only contains the generalized velocity but not its corresponding generalized coordinate, then its associated conjugate momentum is a constant of motion.
Going back to the system described by (51) where now we consider only one particle for simplicity, under the assumption that both scalar and vector potentials do not depend on the position but are, in fact, constant, namely

$$\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \mathbf{A}}{\partial x} = 0, \tag{60}$$

we have that

$$p_x = m\dot{x} + qA_x \text{ is conserved.} \tag{61}$$

Translation invariance:
Let us now consider an infinitesimal translation of the entire system in some given direction $\hat{\boldsymbol{n}}$, namely $q_j \rightarrow q_j + \hat{n}dq_j$. Also, we will consider only velocity independent potentials, i.e. $V = V(\{q_i\})$. In such case we have

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = \frac{d}{dt}p_j = \dot{p}_j = -\frac{\partial V}{\partial q_j} = Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}, \tag{62}$$

We want to show that the above is nothing but the equation of motion for the total linear momentum. In particular we have

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\mathbf{r}_i^{trans} - \mathbf{r}_i}{dq_j} = \frac{dq_j}{dq_j}\hat{n} = \hat{n}, \tag{63}$$

where $\mathbf{r}_i^{trans} = \mathbf{r}(q_i + \hat{n}dq_i)$. The generalized force thus reads

$$Q_j = \sum_i \mathbf{F}_i \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \mathbf{F}_{tot}, \tag{64}$$

which is nothing but the component of the total force in the direction of the translation.
What is more, we also have that

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial \dot{q}_j} = \hat{\boldsymbol{n}} \cdot \sum_i m_i \dot{\boldsymbol{r}}_i, \tag{65}$$

which is the total linear momentum in the $\hat{n}$ direction. Hence, from equation (62) we have

$$\dot{p}_j = Q_j. \tag{66}$$

From this analysis and the considerations made in the previous section we should be able to recover the conservation of momentum in a given direction provided that the translation coordinate q_j is cyclic, namely

$$-\frac{\partial V}{\partial q_j} = Q_j = 0. \tag{67}$$

Immediately we see that this is indeed the case because we now have

$$\dot{p}_j = Q_j = 0 \quad \to \quad p_j \text{ is conserved.} \tag{68}$$

In a similar manner we could verify the conservation of angular momentum when the dq_j correspond to a rotation of the system and q_j is cyclic.

Energy function and conservation of energy

At the beginning of the section we have boldly stated that the conservation of energy is a byproduct of the time independence of the system but we haven't provided any proof of such statement. We plan on doing that now, however, we will see that time independence only leads to energy conservation under special circumstances.
Let us have

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(q_i, \dot{q}_i, t) &= \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial t} + \sum_j \frac{\partial \mathcal{L}}{\partial q_j}\frac{\partial q_j}{\partial t} + \frac{\partial \mathcal{L}}{\partial t} \\ &= \frac{d}{dt}\underbrace{\Big[\sum_j \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L}\Big]}_{h} + \frac{\partial \mathcal{L}}{\partial t}, \end{aligned} \tag{69}$$

or equivalently

$$\boxed{\frac{d}{dt}h = -\frac{\partial \mathcal{L}}{\partial t}}. \tag{70}$$

This automatically means that, if the Lagrangian is time independent then what is conserved is the quantity h, which is not necessarily the energy, namely

$$h \neq T + V \quad \text{in general.} \tag{71}$$

Let us firstly consider the case for which h is indeed the energy:

$$h = \sum_j \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L} = \sum_j p_j \dot{q}_j - \mathcal{L} = 2T - (T - V) = T + V. \tag{72}$$

It seems we have been general and haven't imposed anything, however we did impose that

$$2T = \sum_j p_j \dot{q}_j, \tag{73}$$

namely that the quadratic term in the velocities in the Lagrangian expression only corresponds to the kinetic energy. Let us be more precise and cast a general Lagrangian as a polynomial of order 2 in the velocities, namely

$$\mathcal{L} = \mathcal{L}_0(q,t) + \mathcal{L}_1(q,t)\dot{q} + \mathcal{L}_2(q,t)\dot{q}^2 \tag{74}$$

In this case we thus have

$$h = \frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} = 2\mathcal{L}_2\dot{q}^2 + \mathcal{L}_1\dot{q} - \mathcal{L}_2\dot{q}^2 - \mathcal{L}_1\dot{q} - \mathcal{L}_0 = \mathcal{L}_2\dot{q}^2 - \mathcal{L}_0, \tag{75}$$

which is the total energy only if we regard the following to hold

$$\begin{aligned} \mathcal{L}_2\dot{q}^2 &= T \\ -\mathcal{L}_0 &= V. \end{aligned} \tag{76}$$

In other words, the quadratic term only corresponds to the kinetic energy of the generalized coordinates and the potential is a function of the generalized coordinates only and not their time derivative.

A trivial example where h is indeed the energy of the system given by the following Lagrangian

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x,y,z), \tag{77}$$

so that

$$h = \sum_j \dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L} = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x,y,z) = T + V. \tag{78}$$

A counter example where h is not the total energy can be seen if we consider a free particle in a rotating system of coordinates with angular velocity ω in 2 dimensions. In this case we have

$$\mathcal{L} = T = \frac{1}{2}m[(\dot{q}_1 - \omega q_2)^2 + (\dot{q}_2 + \omega q_1)^2], \tag{79}$$

with

$$\begin{aligned} q_1 &= x\cos\omega t + y\sin\omega t \\ q_2 &= -x\sin\omega t + y\cos\omega t. \end{aligned} \tag{80}$$

The Lagrangian is a quadratic expression in the velocities but it is not a homogeneous polynomial in such variables, indeed we also have quadratic terms in the generalized coordinates. This leads to

$$\begin{aligned} p_1 &= \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = m(\dot{q}_1 - \omega q_2) \\ p_2 &= \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = m(\dot{q}_2 + \omega q_1) \end{aligned} \tag{81}$$

and consequently to

$$h = p_1\dot{q}_1 + p_2\dot{q}_2 - \mathcal{L} = m(\dot{q}_1^2 - \omega q_2\dot{q}_1 + \dot{q}_2^2 + \omega q_1\dot{q}_2) - T \neq T. \tag{82}$$

As a matter of fact it is

$$h = T - \omega q_2\dot{q}_1 + \omega q_1\dot{q}_2, \tag{83}$$

which clearly reduces to the total energy (in this case only kinetic) when $\omega \to 0$. In such case we simply return to the non rotating system where the generalized coordinates are simply the usual Cartesian ones.

Let us look at some other examples and verify what is conserved.

1. Two particles in a relative potential:

$$\mathcal{L} = \frac{1}{2}M\dot{\boldsymbol{R}}^{\mathbf{2}} + \frac{1}{2}\mu\dot{\boldsymbol{r}}^{\mathbf{2}} - V(\mathbf{r}), \tag{84}$$

where

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \qquad \mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1, \tag{85}$$

and

$$M \equiv m_1 + m_2, \qquad \mu \equiv \frac{m_1 m_2}{M}. \tag{86}$$

The Lagrangian is separable into two parts: the center of mass and the relative motion. We have 6 generalized coordinates, three of which do not appear in the Lagrangian expression, namely the X, Y and Z coordinates of the center of mass of the two body system. These only means that their associated conjugate momenta are conserved, namely

$$p_X = M\dot{X},\, p_Y = M\dot{Y},\, p_Z = M\dot{Z} \quad \text{are conserved.} \tag{87}$$

What is more, the Lagrangian is clearly quadratic in the generalized velocities only and thus

$$h = T + V \qquad \text{is conserved.} \tag{88}$$

2. Mass on a central spring:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{k}{2}(r - L)^2, \tag{89}$$

where r is the distance of the mass with respect to the origin and ϕ the angle with respect to the x-axis. Clearly ϕ is a cyclic coordinate which entails that

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi} \quad \text{is conserved.} \tag{90}$$

On the other hand we have

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}, \tag{91}$$

which is not conserved, however it is

$$h = p_r \dot{r} + p_\phi \dot{\phi} - \mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{k}{2}(r-L)^2 = E, \tag{92}$$

thus the energy is indeed conserved.
Another way of expressing it is given by

$$E = \underbrace{\frac{1}{2}m\dot{r}^2}_{T_{eff}} + \underbrace{\frac{p_\phi^2}{2mr^2} + \frac{k}{2}(r-L)^2}_{V_{eff}}, \tag{93}$$

where we have split it into two effective parts, namely the only quadratic term in the radial velocity as the effective kinetic energy $T_{eff}(\dot{r})$ and an effective potential $V_{eff}(r)$, the rest, which only depends on the distance to the origin. Therefore the equation of motion reads

$$m\ddot{r} = -\frac{\partial V_{eff}}{\partial r} = \frac{p_\phi^2}{mr^3} - k(r-L). \tag{94}$$

Its stationary solution (i.e. $\ddot{r} = 0$) entails

$$\frac{p_\phi^2}{mr^3} = k(r-L). \tag{95}$$

Let r_0 be the stationary solution, we can expand the effective potential around this point and find the frequency of small oscillations, namely it is

$$V_{eff}(r) = V_{eff}(r_0) + \left.\frac{\partial V_{eff}}{\partial r}\right|_{r_0}(r-r_0) + \frac{1}{2}\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r_0}(r-r_0)^2 + \dots \tag{96}$$

therefore we end up with

$$\omega^2 = \frac{1}{m}\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r_0}. \tag{97}$$