Classical Mechanics Notes Week 4

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Tuesday, February 9^{th}

1 Central Force Problems

We start by reducing the 2 body problem to a 1 body problem by choosing to work in the center of mass reference frame.

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2} \tag{1}$$

Now the generalized potential does not depend on the center of mass coordinates $U \neq U(\vec{R})$ but instead only on the internal coordinates $U = U(\vec{r_i}, \dot{\vec{r_i}}, ...)$. Since the Lagrangian now only depends on $\dot{\vec{R}}$ this means that $\vec{R}$ is cyclic and $\vec{P}$ is conserved. We also must replace the mass in of the internal kinetic energy with the reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$. Note an easy way of quickly "deriving" this reduced mass is to just think of the dimensions and the fact that it must be symmetric in m_1 and m_2.

Taking the potential to be a conservative central force so $V = V(r)$ and $\vec{F} = -\frac{\partial V}{\partial r}\hat{r}$. Since the potential is spherically symmetric any rotation will leave the equation of motion unchanged. This leads to the angular momentum being conserved. There are 3 constraints because angular momentum is a vector

$$\vec{L} = \vec{r} \times \vec{p} \tag{2}$$

This means that $\vec{r}$ will always be in a plane perpendicular to $\vec{L}$. Choosing a special case where $\vec{r} \parallel \vec{p}$ means that $\vec{L} = 0$ yielding straight line motion.

If instead we choose $\hat{z} \parallel \vec{L}$ then $\phi = \frac{pi}{2}$. This uses two of our three constraints. Note that in physics we use the notation of (r, θ, ϕ) where the transformation equations of spherical to Cartesian are as follows.

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{3}$$

Now writing out the Lagrangian $\mathcal{L} = T - V$

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{4}$$

where θ is cyclic since $\mathcal{L}$ only depends on $\dot{\theta}$. This means that p_θ is conserved and is equal to a constant we will define as ℓ_z.

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = \ell_z \tag{5}$$

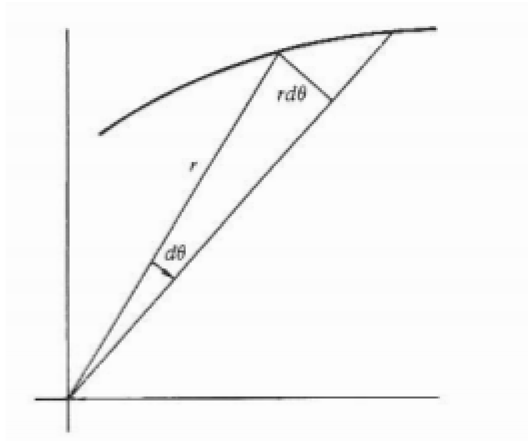


Figure 1: the area swept out by the radius vector in the time dt

With angular momentum being a constant the motion is completely planar. Using equation (5) we can deduce an equation for the area being swept $dA = \frac{1}{2}r^2 d\theta$. Dividing both side by dt we get an equation for areal velocity.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{6}$$

The areal velocity is constant because the angular momentum is constant. This is the proof of Kepler's second law of planetary motion: The radius vector sweeps out equal areas in equal times. This law applies to any potential $V(r)$ that is central. We can also write our angular momentum in the form $p_\theta = \mu|\vec{r} \times \dot{\vec{r}}|$.

Now solving the Euler Lagrange equation for r

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = \mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{7}$$

Changing the spatial derivative of the potential into a force and substituting the constant angular momentum equation (5) in as $\dot{\theta}^2 = \frac{\ell_z^2}{\mu^2 r^2}$, this term is the centripetal force.

$$\mu\ddot{r} - \frac{\ell_z^2}{\mu r^3} = f(r) \tag{8}$$

2 Energy Conservation

In this section we will prove the conservation of energy in three different ways.

1. Noether's Theorem: According to Noether's theorem the time independence of the potential $V \neq V(t)$, leads to energy conservation.

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \tag{9}$$

2. Manipulation of the equation of motion: By rewriting the equation of motion, equation(7), into a derivative with respect to r of both the potential and the centripetal force we come to the equation.

$$\mu\ddot{r} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(V + \frac{1}{2}\frac{\ell_z^2}{\mu r^2}\right) \tag{10}$$

Next multiplying both sides of the equation by $\dot{r}$. the left side of the equation becomes

$$\mu\ddot{r}\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mu\dot{r}^2\right) \tag{11}$$

The right hand side of the equation becomes

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(V + \frac{1}{2}\frac{\ell_z^2}{\mu r^2}\right)\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(V + \frac{1}{2}\frac{\ell_z^2}{\mu r^2}\right) \tag{12}$$

Now putting it all back together yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(V + \frac{1}{2}\frac{\ell_z^2}{\mu r^2}\right) \tag{13}$$

Adding the right hand side over to the left it becomes apparent that the time derivative of the energy is equal to zero and therefore the energy is conserved.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mu\dot{r}^2 + V + \frac{1}{2}\frac{\ell_z^2}{\mu r^2}\right) = 0 \tag{14}$$

3. Energy function h: We use the definition of the energy function

$$h = \sum_i p_i \dot{q}_i - \mathcal{L} \tag{15}$$

Applying the sum over both independent variables r and θ yields the function

$$h = (\mu\dot{r})\dot{r} + \mu r^2\dot{\theta}^2 - \left[\frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V\right] \tag{16}$$

We can see that the kinetic energy terms and the centripetal force terms subtract quite nicely.

$$h = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V = E \tag{17}$$

Having thoroughly proved energy conservation, we can now solve for $\dot{r}$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \left[\frac{2}{\mu}\left(E - V - \frac{\ell_z^2}{2\mu r^2}\right)\right]^{1/2} \tag{18}$$

Treating this as a differential equation and separating variables we can solve for time by integrating over r to get a function $t(r)$

$$t(r) = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{\ell_z^2}{2\mu r^2}\right)}} \tag{19}$$

This integral can be quite difficult so it is preferable to use books of known integrals to look up similar solutions. After solving the integral, inverting the time function finally yields $r(t)$.

3 Orbits

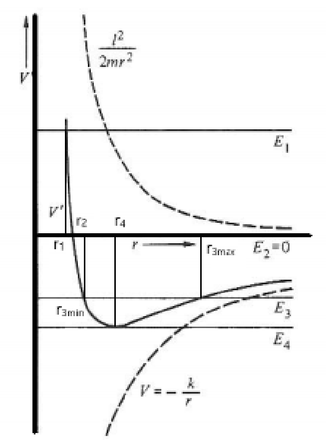
This section focuses on getting as much information from the previous problem without having to integrate. We start by creating a fictitious force made up from the actual force and the centripetal force.

$$f' = f + \frac{\ell_z^2}{\mu r^3} \tag{20}$$

We can then deduce a fictitious potential from this force.

$$V' = V + \frac{1}{2}\frac{\ell_z^2}{\mu r^2} \tag{21}$$

Plotting this fictitious potential gives us a hint as to what orbits can be seen at different energy levels. We now take a look at 4 cases of different energy levels.



The fictitious potential plotted in bold vs r. The upper bound dotted line is the angular momentum centripetal barrier. The lower bound is the potential $V = -\frac{k}{r}$. We require that $|V|$ decreases slower than $\frac{1}{r^2}$ as $r \to \infty$ and $|V|$ increases slower than $\frac{1}{r^2}$ as $r \to 0$

1. $E_1 > 0$ and $r > r_1$: We can rewrite our energy as $E_1 = \frac{1}{2}\mu\dot{r}^2$ as $r \to \infty$. Also note if $V \sim \frac{-k}{r}$ then we get an orbit that is a hyperbola.

2. $E_2 = 0$ and $r > r_2$: for this energy as r goes to infinity we get $\frac{1}{2}\mu\dot{r}_\infty^2 = 0$. Also note if $V \sim \frac{-k}{r}$ then we get an orbit that is a parabola.

3. $E_3 < 0$ and $r_3^{min} < r < r_3^{max}$:the orbit is bounded. For some incidents the orbit is closed, meaning the orbit repeats itself. Also note if $V \sim \frac{-k}{r}$ then we get an orbit that is an ellipse.

4. $E_4 = V'_{min}$ and $r = r_4$:the orbit is will always be a circle since their is only one possible value of r.

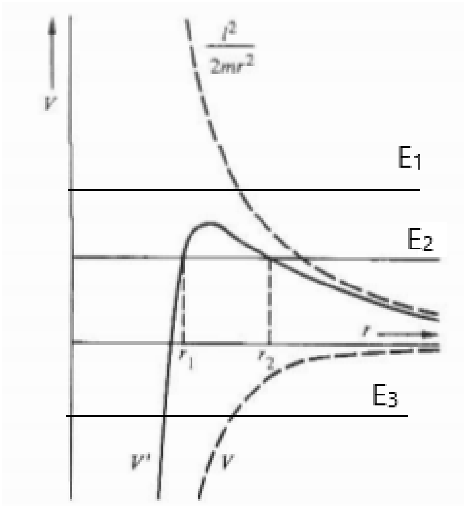
Notice that we never specified the value of l_z. As l_z increases the obit energies and the values at min change but the shapes do not.

Now we consider a couple weird cases. First we have the potential $V(r) = \frac{-k}{r^3}$, figure (a) below. This potential fails both the limit tests given above. We consider three energy ranges. The first is when $E_1 > V'$ where r is unbounded. The second is when $E > 0$ but it is bounded. In this case $r < r_1$ or $r > r_2$

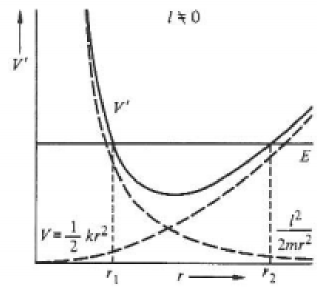
Next we consider the Isotropic Harmonic Oscillator, figure (b) below. Using the harmonic oscillator force $f = -kr$ and potential $V = \frac{1}{2}kr^2$. If $\ell = 0$ then we just have $V' = V$ 1D motion. If $\ell \neq 0$ then we get $V' = \frac{\ell_z^2}{2\mu r^2} + \frac{k}{2}r^2$. The equilibrium is found when $\frac{\partial V'}{\partial r} = \frac{-\ell_z^2}{\mu r^3} + kr = 0$. Solving for r gives the equilibrium radius of $r_{eq}^4 = \frac{\ell_z^2}{\mu k}$. Plugging the radius at equilibrium into the potential gives us the potential at equilibrium.

$$V'_{eq} = \frac{\ell_z^2\sqrt{\mu k}}{2\mu\ell_z} + \frac{k}{2}\frac{\ell_z}{\sqrt{\mu k}} = \ell_z\sqrt{\frac{k}{\mu}} = \ell_z\omega \tag{22}$$

We can also see that since the energy is at V_{min} there is no kinetic energy $T'_{eq} = \frac{1}{2}m\dot{r}^2 = 0$.



(a) The $\frac{1}{r^3}$ Potential



(b) The Isotropic Harmonic Oscillator Potential

Now we can analyze the period of the harmonic oscillator. by taking the relation that the period is equal to the area divide by the areal velocity and using the areal velocity we found earlier we arrive at the equation.

$$Period = \frac{\pi r_{eq}^2}{\frac{\ell_z}{2\mu}} = \frac{2\pi\mu r_{eq}^2}{\ell_z} \tag{23}$$

Using the definition of then angular momentum from earlier $\ell_z = \sqrt{\mu k} r_{eq}^2$. This shows that ℓ_z is proportional to r.

$$Period = \frac{2\pi\mu r_{eq}^2}{\sqrt{\mu k} r_{eq}^2} = 2\pi\sqrt{\frac{\mu}{k}} = \frac{2\pi}{\omega} \tag{24}$$

If we look at the Cartesian solution $\mathcal{L} = \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2) + \frac{k}{2}(x^2 + y^2)$ Where we solve for x and y and get

$$\begin{aligned} x &= A_x \cos\omega t + \phi_x \\ y &= A_y \sin\omega t + \phi_y \end{aligned} \tag{25}$$

At equilibrium we have $A_x = A_y$ and $\phi_x = \phi_y$.

Now taking into consideration small oscillations around the equilibrium we expand the fictitious potential into a Taylor series.

$$V'(r) = V'(r_{eq}) + \frac{\partial v'}{\partial r}\delta r + \frac{1}{2}\frac{\partial^2 V'}{\partial r^2}\bigg|_{r_{eq}} \delta r^2 \tag{26}$$

The first derivative term we already know to be zero and the first term is a constant. Taking the second derivative we see that

$$V'(r) = C + \frac{1}{2}\left(\frac{3\ell^2}{\mu r_{eq}^4} + k\right)\delta r^2 \tag{27}$$

Once again using the definition $\ell_z = \sqrt{\mu k} r_{eq}^2$

$$V'(r) = C + \frac{1}{2}(3k + k)\delta r^2 = C + \frac{1}{2}(4k)\delta r^2 \tag{28}$$

This yields an frequency of small oscillations of $\omega_{osc} = \sqrt{\frac{k'}{\mu}} = \sqrt{\frac{4k}{\mu}} = 2\omega$. Since 2 is rational we confirm that this is a closed orbit.

Finally addressing turning points at the minimum and maximum r. Since we are only looking at extreme values of r we known that $\dot{r} = 0$. Therefore our energy equation becomes.

$$E = \frac{\ell_z^2}{2\mu r^2} + \frac{k}{2} r^2 \tag{29}$$

Multiplying through by $r\frac{2r^2}{k}$ yields a quadratic equation in r^2

$$r^4 - \frac{2E}{k} r^2 + \frac{\ell_z^2}{\mu k} = 0 \tag{30}$$

Using the quadratic equation we an solve for r^2

$$r^2_{max,min} = \frac{E}{k} \pm \left[\frac{E^2}{k^2} - \frac{\ell_z^2}{\mu^2 k^2} \right]^{1/2} \tag{31}$$

February Thursday, February 11^{th}

4 Virial Theorem

The Virial theorem is another property of central force motion and is a statistical statement about T and V. We define a function $G = \sum_i \vec{p}_i \cdot \vec{r}_i$. Taking the time derivative we arrive at

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \dot{\vec{r}}_i \tag{32}$$

Where the first term on the right hand side of the equation is $\vec{F}_i \cdot \vec{r}_i$ and the second term is $2T = \sum_i m v_i^2$. Therefore we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \vec{p}_i \cdot \vec{r}_i \right) = \sum_i \vec{F}_i \cdot \vec{r}_i + 2T \tag{33}$$

Taking the time average of everything we see that

$$\frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}G}{\mathrm{d}t} dt = \overline{\frac{\mathrm{d}G}{\mathrm{d}t}} = \overline{2T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i} \tag{34}$$

The bar on top of the terms denotes time average. Now solving the integral gives the equation

$$\overline{2T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i} = \frac{1}{\tau}(G(\tau) - G(0)) \tag{35}$$

If the motion is periodic or bounded then ΔG is finite and $\tau \to \infty$. Therefore the right hand side of the equation becomes zero. This leaves us with the Virial Theorem

$$\overline{T} = -\overline{\frac{1}{2}\sum_i \vec{F}_i \cdot \vec{r}_i} \tag{36}$$

If the force is conservative it can be written as $\vec{F} = -\vec{\nabla} V$ and the Virial theorem becomes

$$\overline{T} = \frac{1}{2}\overline{\sum_i \vec{\nabla} V \cdot \vec{r}_i} \tag{37}$$

In the case of a single particle in a central potential $V = V(r)$

$$\overline{T} = \frac{1}{2}\frac{\partial V}{\partial r} r \tag{38}$$

If the potential is of the common form $V = ar^{n+1}$ then $\frac{\partial V}{\partial r} r = (n+1)V$ and we can solve the virial theorem for two familiar cases.

The first case is the harmonic oscillator, where $n = 1$. This yields a virial theorem of $\overline{T} = \frac{1}{2}\overline{V}$.

The second case is for gravity, where $n = -2$. This yields a virial theorem of $\overline{T} = -\frac{1}{2}\overline{V}$. This shows that in order to reach escape velocity one must double the kinetic energy. In other words the escape velocity is equal to $\sqrt{2}$ Circular velocity. This exact theory is actually used in determining galaxy cluster mass. If we know the average kinetic energy from redshift data we can find the mass using the virial theorem.

5 Orbits Part II: The Orbiting

Now we will be using m instead of μ. Everything will still be in terms of reduced mass but we will just represent it by m now. We will also be dropping the z in ℓ_z for convenience. In this section we will work to first find $r(\theta)$ and then find $r(t)$ and $\theta(t)$ later.

We start by considering the conserved angular momentum $\ell = mr^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}$. Separating time and angle we come the the equation $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\ell}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta}$. Now that we have established a relationship between a time derivative and a derivative with respect to θ we can manipulate the equation of motion.

$$m\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{\ell}{mr^3} = f(r) \to \frac{\ell}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\ell}{mr^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right) - \frac{\ell}{mr^3} = f(r) \tag{39}$$

We define $U = \frac{1}{r}$ and $f = -\frac{\partial V}{\partial r} = -\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{\mathrm{d}V}{\mathrm{d}U}\frac{\mathrm{d}U}{\mathrm{d}r} = U^2\frac{\mathrm{d}V}{\mathrm{d}U}$. Now the complex differential term of our equation of motion becomes

$$\frac{\ell^2}{m}U^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(U^2\frac{\mathrm{d}1/U}{\mathrm{d}\theta}\right) = \frac{\ell^2}{m}U^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left[U^2\left(-U^{-2}\frac{\mathrm{d}V}{\mathrm{d}\theta}\right)\right] = -\frac{\ell^2 U^2}{m}\frac{\mathrm{d}^2 V}{\mathrm{d}\theta^2} \tag{40}$$

Plugging this back into the full equation of motion

$$\frac{\mathrm{d}^2 U}{\mathrm{d}\theta^2} + U = -\frac{m}{\ell^2}\frac{\mathrm{d}}{\mathrm{d}U}V(1/U) \tag{41}$$

Notice how this equation of motion is symmetric in θ

If we chose $\theta = 0$ at a turning point, where r is a max or min, then $U_0 = U(0)$ and $\left.\frac{\mathrm{d}U}{\mathrm{d}\theta}\right|_0 = 0$. Returning now to our integral to solve for time, equation(19), and using our time θ relation we derived earlier.

$$\theta = \theta_0 - \int_{U_0}^{U} \frac{dU}{\left[\frac{2m}{\ell^2}(E - V) - U^2\right]^{1/2}} \tag{42}$$

If the potential is the familiar $V = ar^{n+1}$ then for $n = 1, -2, -3$ we get trig function solutions and for $n = 5, 3, 0, 7$ we get elliptical functions as special cases of generalized hypergeometric functions.

6 Closed Orbits (Bertrand's Theorem)

Now we deal with the problem of finding out which orbits are closed. We start by looking at only small perturbations $U = U_0 + G\cos\beta\theta$. If β is a rational number then the orbit will be closed.

The treatment of big perturbations is a bit more complex. They only result in closed orbits only for r^2 and r^-1 potentials which are the familiar Harmonic Oscillator and gravity. Since orbits are closed only for exponent exactly -1, this shows that the gravitational force $\propto r^-2$ exactly which means that space has exactly 3 dimensions (not 2.95 or 3.01).

7 The Kepler Problem

For gravity we have $F = \frac{-k}{r^2}$ and $V = -\frac{k}{r}$. Plugging this back into our integral for theta

$$\theta = \theta' - \int_{U_0}^{U} \frac{dU}{\left[\frac{2mE}{\ell^2} + \frac{2mk}{\ell}U - U^2\right]^{1/2}} \tag{43}$$

Looking up a similar integral we see that $\int \frac{dx}{(\alpha+\beta x+\gamma x^2)^{1/2}} = \frac{1}{\sqrt{-\gamma}}\cos^{-1}(\frac{-\beta+2\gamma\alpha}{\sqrt{q}})$ where $q = \beta^2 - 4\alpha\gamma$. Now using this on our equation we see that

$$\theta = \theta' - \cos^{-1}\left(\frac{\ell^2 U/mk - 1}{\left[1 + \frac{2E\ell^2}{mk^2}\right]^{1/2}}\right) \tag{44}$$

Inverting and solving for U

$$U = \frac{1}{r} = \frac{mk}{\ell^2}\left(1 + \left[1 + \frac{2E\ell^2}{mk^2}\right]^{1/2}\right) cos(\theta - \theta') \tag{45}$$

where θ' is the turning angle. E,ℓ, and θ' are three of the four constraints of motion, or "integration". This equation matches that of a conic with one focus at the origin $\frac{1}{r} = C\left[1 + e\cos(\theta - \theta')\right]$. Relating this to our equation above gives the eccentricity $e = \left[1 + \frac{2E\ell^2}{mk^2}\right]^{1/2}$. Referring back to the beginning of the orbits section we have

$$\begin{aligned}
E > 0 \rightarrow e > 1 &\qquad \textit{hyperbolic} \\
E = 0 \rightarrow e = 1 &\qquad \textit{parabola} \\
E < 1 \rightarrow e < 1 &\qquad \textit{ellipse} \\
E = \frac{-mk^2}{2\ell^2} \rightarrow e = 0 &\qquad \textit{circle}
\end{aligned} \tag{46}$$

For a circular orbit we can apply the virial theorem and since the radius never changes the time average is unnecessary $E = T + V = -\frac{V}{2} + V$. Therefore $E = -\frac{k}{2r_0}$ and we can see the relationship $\frac{k}{r_0^2} = \frac{\ell^2}{mr_0^3}$ and therefore $r_0 = \frac{\ell^2}{mk}$. So the energy can be written as $E = -\frac{mk}{2\ell^2}$ Now we can see the relationships between r, ℓ, and E. If r increases, ℓ increases, and E decreases.

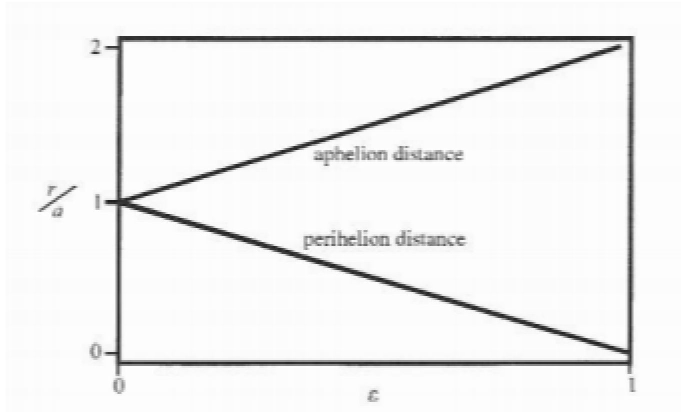


Figure 3: The relation between aphelion, perihelion, and eccentricity

Moving on to working with other conics we introduce some new variables. First, a is defined as the semimajor axis and is defined as $\frac{1}{2}(r_1 + r_2)$, where r_1 is the perihelion and r_2 is the aphelion. At the points of perihelion and aphelion the velocity is zero. We once again return to the equation of motion $E = \frac{\ell^2}{2mr^2} - \frac{k}{r}$. We can easily change this into a quadratic equation for r: $r^2 + \frac{k}{E} - \frac{\ell^2}{2mE} = 0$. Solving the quadratic formula yields

$$r_{1,2} = \frac{-\frac{k}{E} \pm \sqrt{\frac{k^2}{E^2} - 4\frac{\ell^2}{2mE}}}{2} \tag{47}$$

Remember that the energy of bound orbits is negative so this wont yield any complex solutions. We can also see from this equation that $\frac{1}{2}(r_1 + r_2) = -\frac{k}{2E} = a$ and this is completely independent of ℓ. Substituting this energy into the eccentricity equation

$$e = \left[1 - \frac{\ell^2}{mka}\right]^{1/2} \tag{48}$$

For $0 < e < 1$ we get an ellipse. we can use this to solve for the radius now.

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \theta')} \tag{49}$$

If $cos(\theta) = 1$ then $r = a(1 - e)$ and if $cos(\theta) = -1$ then $r = a(1 + e)$.This is demonstrated in figure 4 above.

Looking at turning points, which are perihelion and aphelion, the radial velocity is zero. This means that at perihelion v_θ is maximum and at aphelion v_θ is minimum. This difference in velocity is a direct example of Kepler's second law.

Returning now to the time integral of before and using θ' as a turning point.

$$t = \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr}{\left[\frac{k}{r} - \frac{\ell^2}{2mr^2} + E\right]^{1/2}} = \frac{\ell^3}{mk^2} \int_{\theta_0}^{\theta} \frac{d\theta}{\left[1 + e\cos(\theta - \theta')\right]^2} \tag{50}$$

From these equations we can solve for $t(r)$ and $t(\theta)$, which end up being quite ugly functions. Inverting these functions we can get $r(t)$ and $\theta(t)$, which are even uglier functions.

Instead of trying to solve these ugly equations instead we can analyze special cases. First we will look at the case of a parabola, where $e = 1$. Choosing $\theta' = 0$ and using the trigonometric identity $1 + \cos(\theta) = \frac{1}{2}cos^2\left(\frac{\theta}{2}\right)$ we get simplified integral

$$t = \frac{\ell^3}{4mk^2}\int_{\theta_0}^{\theta} \sec^4\left(\frac{\theta}{2}\right) d\theta \tag{51}$$

This integral can be solve with some trig substitution. Setting $x = \tan\left(\frac{\theta}{2}\right)$ and therefore $dx = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) d\theta$ the integral reduces to

$$t = \frac{\ell^3}{2mk^2}\int_{0}^{\tan\left(\frac{\theta}{2}\right)} (1 + x^2)dx \tag{52}$$

Yielding the equation

$$t = \frac{\ell^3}{2mk^2}\left[\tan\left(\frac{\theta}{2}\right) + \frac{1}{3}\tan^3\left(\frac{\theta}{2}\right)\right] \tag{53}$$

where the range of θ is given by $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$ and the range of time is $-\infty < t < \infty$.

Next we look at the special case of the ellipse, when $e < 1$. We start by defining the variable ψ so that $r = a(1 - e\cos(\psi))$. At perihelion $\psi = \theta = 0$ and at aphelion $\psi = \theta = \pi$. Integrating over ψ now becomes simple

$$t = \sqrt{\frac{ma^3}{k}}\int_0^{\psi} (1 - e\cos(\psi))d\psi \tag{54}$$

This integral provides the expression of the period if we consider ψ to be over a full orbit $0 \rightarrow 2\pi$. solving then gives the period of the orbit.

$$\tau = 2\pi a^3\sqrt{\frac{m}{k}} \tag{55}$$

And we see the oh so important conclusion of Kepler's second law one more time $\tau^2 \propto a^3$.

Applying this to our own solar system we see that in the relationship of Kepler's second law the constant multiplied by a^3 is given by the sum of mass of the sun and the mass of the planet by which you are calculating the orbit off. Kepler's proportionality works perfectly if we ignore the mass of the planet and only consider the mass of the sun. We consider the relative mass of the planet as the error in Kepler's relationship. For Jupiter this is the most apparent with $\frac{M_{jupiter}}{M_{sun}} = 10^{-3}$.

References

[1] Goldstein, Herbert, et al. Classical Mechanics. Pearson, 2014.