Classical Mechanics 603
Lecture Notes Week 6
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1 Rigid body motion

When studying the motion of a rigid body, we start by recalling that a rigid body definition is that the interparticle distances $|\vec{r}_{12}|$, $|\vec{r}_{13}|$,..., are fixed. By fixing those distances, we can reduce the number of constraints to $\frac{N(N-1)}{2}$. Likewise, we can obtain the six necessary coordinates in the following way: We start by establishing the position of one of the reference points, for which three coordinates are given. Later the second point is specified with only two coordinates, and finally, the third point only has one.

Hence, we can write the space coordinate system as $\hat{i}$, $\hat{j}$ and $\hat{k}$, and the body coordinate system as $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$. Therefore, the direction of the cosines can be written as:

$$\begin{aligned} \cos\theta_{11} &= \hat{i}\cdot\hat{i}', \\ \cos\theta_{12} &= \hat{j}\cdot\hat{i}', \\ \cos\theta_{13} &= \hat{k}\cdot\hat{i}', \\ \cos\theta_{21} &= \hat{i}\cdot\hat{j}', \\ &.... \end{aligned} \tag{1}$$

We can also express the unit vectors of the primed system in terms of the cosines as:

$$\begin{aligned} \hat{i}' &= \cos\theta_{11}\hat{i} + \cos\theta_{12}\hat{j} + \cos\theta_{13}\hat{k}, \\ \hat{j}' &= \cos\theta_{21}\hat{i} + \cos\theta_{22}\hat{j} + \cos\theta_{23}\hat{k}, \\ \hat{k}' &= \cos\theta_{31}\hat{i} + \cos\theta_{32}\hat{j} + \cos\theta_{33}\hat{k}. \end{aligned} \tag{2}$$

Therefore, we can write $\hat{i}$, $\hat{j}$, $\hat{k}$ in terms of the primed axis and hence we obtain:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x'\hat{i}' + y'\hat{j}' + z\hat{k}', \tag{3}$$

which allows us to write x', y', and z' in terms of x, y, and z and vice versa.

If the primed axis is fixed with respect to the body, then the direction given by the cosines will change as the object rotates. From this, we can see that only 3 directions are now independent. This follows from the orthogonality relations:

$$\begin{aligned} \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \hat{i}'\cdot\hat{j}' = \hat{j}'\cdot\hat{k}' = \hat{k}'\cdot\hat{i}' = 0, \\ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = \hat{i}'\cdot\hat{i}' = \hat{j}'\cdot\hat{j}' = \hat{k}'\cdot\hat{k}' = 1. \end{aligned} \tag{4}$$

Now, to better study the properties of the directions, from now we will make the following change in the notation:

$$\begin{aligned} x &\to x_1, \\ y &\to x_2, \\ z &\to x_3, \end{aligned} \tag{5}$$

and

$$\cos\theta_{ij} \to a_{ij}. \tag{6}$$

Hence we can write:

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x'_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x'_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3. \end{aligned} \tag{7}$$

To simplify this expression, we can write

$$x'_i = a_{ij}x_i, \tag{8}$$

where the Einstein sum convention is implied. Knowing $\vec{x'} \cdot \vec{x'} = \vec{x} \cdot \vec{x}$, then $\vec{x'}^T \vec{x'} = \vec{x'}^T \overleftrightarrow{\mathbb{A}}^T \overleftrightarrow{\mathbb{A}} \vec{x'}$, where $\overleftrightarrow{\mathbb{A}}^T \overleftrightarrow{\mathbb{A}}$ would be the identity matrix $\overleftrightarrow{\mathbb{1}}$, and

$$\vec{x'}^T = \vec{x}^T \overleftrightarrow{\mathbb{A}}^T. \tag{9}$$

Now, the matrix of transformation $\overleftrightarrow{\mathbb{A}}$ can then be written as:

$$\overleftrightarrow{\mathbb{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{10}$$

with the connection between the elements $a_{ij}a_{ik} = \delta_{jk}$, where $j, k = 1, 2, 3$.

Example

Consider the motion of an object in a plane , with the rotation around the z-axis. The transformation matrix reduces then to:

$$\overleftrightarrow{\mathbb{A}} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

where the four elements are connected by the orthogonality condition $a_{ij}a_{ik} = \delta_{jk}$, with $j, k = 1, 2$, and hence, instead of the apparent 4 constrains we have only 3 (since $a_{12} = a_{21}$), and one degree of freedom.

The rotation is specified by the rotation angle ϕ. The transformation equations expressed in terms of ϕ then becomes:

$$\begin{aligned} x'_1 &= x_1\cos\phi + x_2\sin\phi, \\ x'_2 &= -x_1\sin\phi + x_2\cos\phi, \\ x'_3 &= x_3. \end{aligned} \tag{12}$$

Now, the rotation interpretation can differ as follows: If the transformation is the rotation of $\vec{r}$ by $-\phi$, then is named active; If on the other hand, the transformation is seen as the rotation of the axis by $+\phi$, then can be considered to be passive.

1.1 Matrix Properties

Being $\overleftrightarrow{\mathbb{A}}$ the matrix transformation, the operation that changes $\vec{r}'$ back to $\vec{r}$ can be called the transformation inverse and can be defined as $\overleftrightarrow{\mathbb{A}}^{-1}$. Therefore, we can show that $\overleftrightarrow{\mathbb{A}}\,\overleftrightarrow{\mathbb{A}}^{-1} = \overleftrightarrow{\mathbb{A}}\,\overleftrightarrow{\mathbb{A}}^{T} = \overleftrightarrow{\mathbb{1}}$.

On the other hand, the properties for the determinant for the rotations would be:

- $|\overleftrightarrow{\mathbb{A}}\,\overleftrightarrow{\mathbb{B}}| = |\overleftrightarrow{\mathbb{A}}| \cdot |\overleftrightarrow{\mathbb{B}}|$,
- $|\overleftrightarrow{\mathbb{A}}|^2 = |\overleftrightarrow{\mathbb{A}}| \cdot |\overleftrightarrow{\mathbb{A}}^{T}| = |\overleftrightarrow{\mathbb{1}}| = 1$,

which implies that the determinant of an orthogonal matrix can be ± 1.

Now, if $|\overleftrightarrow{\mathbb{A}}| = 1$, this is representing a physical change in the system, that is the case for rotations. On the other hand, if $|\overleftrightarrow{\mathbb{A}}| = -1$then we have the case of the inversion transformation.

1.1.1 Inversion Transformation

We define the transformation $\overleftrightarrow{S}$, with $|\overleftrightarrow{S}| = -1$, as:

$$\overleftrightarrow{S} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -\overleftrightarrow{\mathbb{1}}. \tag{13}$$

This transformation is called an inversion transformation and has the effect of changing the sign of each component or coordinate axes. One method to perform this inversion would be to do a rotation by π and then reflect in the coordinate axis direction. For example, for the z-direction:

$$\overleftrightarrow{S} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{14}$$

A few remarks about the inversion transformation. This transformation never corresponds to a physical displacement, so even when it turns a right-handed coordinate system into a left-handed system this would be seen as a reflection. In addition, this transformation cannot be done infinitesimally.

1.2 Rotation Transformation

Now, in order to describe a rigid body system with three degrees of freedom, we will use the sets of parameters know as the Euler Angles. When carrying out the transformation from a given system to another, we will do it through three successive rotations that would be performed in a specific sequence. Therefore, the Euler angles would be the three different angles of rotation.

We will first rotate around the z-axis by ϕ:

$$\overleftrightarrow{\mathbb{D}} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

then, we rotate around the new axis x' by θ:

$$\overleftrightarrow{\mathbb{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \tag{16}$$

and finally, we rotate around z' by ψ:

$$\overleftrightarrow{\mathbb{B}} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

so that:

$$\begin{aligned} \overleftrightarrow{\mathbb{A}} &= \overleftrightarrow{\mathbb{B}}\overleftrightarrow{\mathbb{C}}\overleftrightarrow{\mathbb{D}}, \\ &= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ &= \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}. \end{aligned} \tag{18}$$

where $\overleftrightarrow{\mathbb{A}}$ would be the general transformation.

Another convention commonly used in engineering applications can be used, where the rotations are named *roll, pitch* and *yaw* and they are taken around certain axes.

Having discussed the motion of a rigid body we can then write Euler's theorem: *The general displacement of a rigid body with one point fixed is a rotation about some axis.* Therefore, for each rotation is possible to find an axis through the fixed point oriented at a particular angle.

Now, a rotation has two characteristics or conditions that must follow. First, it must leave the axis unchanged, and, second, the magnitude of the vector must be unaffected. Hence, we can write the equation corresponding to the eigenvalues of the rotation matrix as:

$$\vec{r}' = \overleftrightarrow{\mathbb{R}}\vec{r} = \lambda\vec{r}, \tag{19}$$

being λ the eigenvalues of the matrix. We must then find the eigenvalues that satisfies the above equation, so the eigenvalue equations can be written as:

$$(\overleftrightarrow{\mathbb{R}} - \overleftrightarrow{\lambda})\vec{r} = 0, \tag{20}$$

which can be written as the characteristic equation:

$$|\overleftrightarrow{\mathbb{R}} - \overleftrightarrow{\lambda}| = 0, \tag{21}$$

that would allow us to find the values of λ. By writing the matrix of λ as:

$$\overleftrightarrow{\lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{22}$$

Thus, this implies

$$\begin{aligned} \overleftrightarrow{\mathbb{R}}\overleftrightarrow{\mathbb{A}} &= \overleftrightarrow{\mathbb{A}}\overleftrightarrow{\lambda}, \\ \overleftrightarrow{\mathbb{A}}^{-1}\overleftrightarrow{\mathbb{R}}\overleftrightarrow{\mathbb{A}} &= \overleftrightarrow{\mathbb{A}}^{-1}\overleftrightarrow{\mathbb{A}}\overleftrightarrow{\lambda}, \\ \overleftrightarrow{\mathbb{A}}^{-1}\overleftrightarrow{\mathbb{R}}\overleftrightarrow{\mathbb{A}} &= \overleftrightarrow{\lambda}, \\ |\overleftrightarrow{\mathbb{A}}^{-1}\overleftrightarrow{\mathbb{R}}\overleftrightarrow{\mathbb{A}}| = |\overleftrightarrow{\lambda}| &= \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{23}$$

Now, we consider the equation

$$(\overleftrightarrow{\mathbb{A}} - \overleftrightarrow{\mathbb{1}})(\overleftrightarrow{\mathbb{A}}^T) = (\overleftrightarrow{\mathbb{1}} - \overleftrightarrow{\mathbb{A}}^T), \tag{24}$$

then, if we take the determinant on both sides, we can write

$$\begin{aligned} |\overleftrightarrow{\mathbb{A}} - \overleftrightarrow{\mathbb{1}}||\overleftrightarrow{\mathbb{A}}^T| &= |\overleftrightarrow{\mathbb{1}} - \overleftrightarrow{\mathbb{A}}^T|, \\ |\overleftrightarrow{\mathbb{A}} - \overleftrightarrow{\mathbb{1}}| &= |\overleftrightarrow{\mathbb{1}} - \overleftrightarrow{\mathbb{A}}|, \end{aligned} \tag{25}$$

since $|-\overleftrightarrow{\mathbb{B}}| = (-1)^n|\overleftrightarrow{\mathbb{B}}|$. Given that we are working on tridimensional space, with $n = 3$, the above equation must hold true for any rotation if

$$|\overleftrightarrow{\mathbb{A}} - \overleftrightarrow{\mathbb{1}}| = 0. \tag{26}$$

Thus $\lambda = +1$ must be an eigenvalue. From here we have three possibilities:

1. $\lambda_1 = \lambda_2 = \lambda_3 = +1$
   In this case the transformation matrix $\overleftrightarrow{\mathbb{R}}$ is then $\overleftrightarrow{1}$.

2. $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = +1$
   This would be a transformation as an inversion in two coordinate axes. Similarly, it is a rotation through the angle π.

3. $\lambda_3 = 1$ and $\lambda_1 = e^{i\phi}$ and $\lambda_2 = e^{-i\phi}$

Hence, a non-trivial real orthogonal matrix has *only one* eigenvalue +1. Now to find the direction $\hat{n}$ we start by choosing $\lambda = 1$ and solving the eigenvalues equation. Then, we use a similarity transform $\overleftrightarrow{\mathbb{B}}^{-1}\overleftrightarrow{\mathbb{A}}\overleftrightarrow{\mathbb{B}}$ such that the new coordinate z' is parallel to $\hat{n}$.

Now, since $\overleftrightarrow{\mathbb{A}}'$ represent a rotation around the z-axis, the matrix has the form:

$$\overleftrightarrow{\mathbb{A}} = \begin{bmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{27}$$

with the trace described as:

$$Tr(\overleftrightarrow{\mathbb{A}}') = 1 + 2\cos\Phi. \tag{28}$$

Since the trace is invariant under a similarity transformation, then

$$Tr(\overleftrightarrow{\mathbb{A}}') = Tr(\overleftrightarrow{\mathbb{A}}) = \sum_i \lambda_i = 1 + e^{i\Phi} + e^{-i\Phi} = 1 + 2\cos\Phi. \tag{29}$$

Hence, it is clear that the cases 1 and 2, with all eigenvalues real, correspond to a special case of case 3 with complex eigenvalues. From this follows that the direction of the rotation axis and the rotation angle are ambiguous. If $\overleftrightarrow{\mathbb{R}}$ is an eigenvector, so is $-\overleftrightarrow{\mathbb{R}}$, so the direction of the rotation axis is not specified.

Also, if Φ satisfies Eq. (28), then also $-\Phi$. Thus we can make a corollary to Euler's theorem, the Chasle's theorem: *The most general displacement of a rigid body is a translation plus a rotation.*

1.2.1 Finite rotations

Given that the coordinate transformation can be carried out through a single rotation about an axis, then we can seek a representation in terms of the angle of rotation and the direction of the axis of rotation. Suppose we have a vector $\vec{r}$ making an angle θ with respect to the axis $\hat{n}$. By performing a clockwise rotation, by an angle Φ, we can then obtain the relation between $\vec{r}$ and $\vec{r}'$, the final position, as:

$$\vec{r}' = \vec{r}\cos\Phi + \hat{n}(\hat{n}\cdot\vec{r})(1-\cos\Phi) + (\vec{r}\times\hat{n})\sin\Phi. \tag{30}$$

This is usually known as the *Rotation formula.*

1.2.2 Infinitesimal rotations

So far we have been working with transformations $\overleftrightarrow{\mathbb{A}}$ and $\overleftrightarrow{\mathbb{B}}$, that must follow the commutative relation:

$$\overleftrightarrow{\mathbb{A}}\overleftrightarrow{\mathbb{B}} \neq \overleftrightarrow{\mathbb{B}}\overleftrightarrow{\mathbb{A}}. \tag{31}$$

Thus, finite rotations cannot be represented by a single vector, however, the same is not true for infinitesimal rotations. An infinitesimal rotation can be described as an orthogonal transformation in which the components of the vector are the same on the coordinate axis and on the primed axis. Therefore we can write the x'_1 component of a vector as:

$$x'_1 = x_1 + \epsilon_{11}x_1 + \epsilon_{12}x_2 + \epsilon_{13}x_3, \tag{32}$$

where the elements ϵ_{ij} are infinitesimal. This expression can be written in a more compact form as:

$$x'_i = x_i + \epsilon_{ij}x_j, \tag{33}$$

or in matrix notation:

$$\vec{x}' = (\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon})\vec{r}. \tag{34}$$

Now, we can easily prove that the sequence of operations is not important for the infinitesimal rotations, since they commute:

$$\begin{aligned} (\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}_1)(\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}_2) &= \overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}_1\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\mathbb{1}}\overleftrightarrow{\epsilon}_2 + \overleftrightarrow{\epsilon}_1\overleftrightarrow{\epsilon}_2, \\ &= \overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}_1 + \overleftrightarrow{\epsilon}_2, \end{aligned} \tag{35}$$

where we neglect the term $\overleftrightarrow{\epsilon}_1\overleftrightarrow{\epsilon}_2$.

Thus, by ignoring the second-order effects, we can write an infinitesimal Euler rotation around the axis z, then x' and then z' as:

$$\overleftrightarrow{\mathbb{A}} = \begin{bmatrix} 1 & (d\phi + d\psi) & 0 \\ -(d\phi + d\psi) & 1 & d\theta \\ 0 & -d\theta & 1 \end{bmatrix}, \tag{36}$$

and

$$d\vec{\Omega} = \hat{i}d\theta + \hat{k}(d\phi + d\psi). \tag{37}$$

Then, if $\overleftrightarrow{\mathbb{A}} = \overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon}$ is the matrix transformation, then the inverse transformation would be:

$$\overleftrightarrow{\mathbb{A}}^{-1} = \overleftrightarrow{\mathbb{1}} - \overleftrightarrow{\epsilon}, \tag{38}$$

and then the product $\overleftrightarrow{\mathbb{A}}\overleftrightarrow{\mathbb{A}}^{-1}$ reduces to:

$$\overleftrightarrow{\mathbb{A}}\overleftrightarrow{\mathbb{A}}^{-1} = (\overleftrightarrow{\mathbb{1}} + \overleftrightarrow{\epsilon})(\overleftrightarrow{\mathbb{1}} - \overleftrightarrow{\epsilon}) = \overleftrightarrow{\mathbb{1}}. \tag{39}$$

In addition, the orthogonality of the matrix transformation implies that the inverse transformation must be equal to the transpose $\overleftrightarrow{\mathbb{A}}^{-1} = \overleftrightarrow{\mathbb{A}}^T$, and thus, the infinitesimal matrix would be antisymmetric $-\overleftrightarrow{\epsilon} = \overleftrightarrow{\epsilon}^T$. Now, because of this we will have only three degrees of freedom, and the matrix of the infinitesimal rotation can be written as:

$$\overleftrightarrow{\epsilon} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}. \tag{40}$$

We can now write the terms of dx_i in terms of $d\vec{\Omega}_i$:

$$d\vec{r} = \vec{r} \times d\vec{\Omega}, \tag{41}$$

such that:

$$\begin{aligned} dx_1 &= x_2 d\Omega_3 - x_3 d\Omega_2, \\ dx_2 &= -x_1 d\Omega_3 + x_3 d\Omega_1, \\ dx_3 &= x_1 d\Omega_2 - x_2 d\Omega_1. \end{aligned} \tag{42}$$

Notice that $d\vec{\Omega}$ refers to a vector of infinitesimal length. To confirm this, we must remember that vectors have certain properties under rotations. If we rotate a vector $\vec{r}$ by $d\phi$ around $\hat{n}$, then $|d\vec{r}| = r \sin\theta d\phi$. Now, since $d\vec{r}$ is orthogonal to $\vec{r}$, then $d\vec{r}$ is also orthogonal to $\hat{n}$. Thus $d\vec{r} = \vec{r} \times \hat{n} d\phi = \vec{r} \times d\vec{\Omega}$, and since $\hat{n}$ is a vector, $d\vec{\Omega}$ is also a vector.

Example

To study the vector nature of $d\vec{\Omega}$, we recall the properties of the parity operator $\mathbb{P}$:

$$\begin{aligned} \mathbb{P}(\text{scalar}) &= \text{scalar}, \\ \mathbb{P}(\text{polar vector}) &= -\text{polar vector}, \\ \mathbb{P}(\text{axial vector}) &= \text{axial vector}, \\ \mathbb{P}(\text{pseudoscalar}) &= -\text{pseudoscalar}. \end{aligned} \tag{43}$$

Now, since $\vec{r}$, $d\vec{r}$ and $\vec{r} \times \hat{n}$ are polar vectors, then $\hat{n}$ and $d\vec{\Omega}$ would be axial vectors.
Recalling the rotation formula for finite rotations, Eq. (30), then we can write $\overleftrightarrow{\epsilon}$ as:

$$\overleftrightarrow{\epsilon} = \begin{bmatrix} 0 & -d\Omega_3 & d\Omega_2 \\ d\Omega_3 & 0 & -d\Omega_1 \\ -d\Omega_2 & d\Omega_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} d\Phi = \overleftrightarrow{\mathbb{N}}^T d\Phi. \tag{44}$$

Thus we can write

$$\frac{d\vec{r}}{d\Phi} = -\overleftrightarrow{\mathbb{N}}\vec{r}, \tag{45}$$

where $\overleftrightarrow{\mathbb{N}} = N_{ij} = \varepsilon_{ijk} n_k$, and ε_{ijk} is the Levi-Civita symbol.
Alternatively, we can write $\overleftrightarrow{\epsilon}$ as

$$\overleftrightarrow{\epsilon} = n_i \overleftrightarrow{\mathbb{M}}_i d\Phi, \tag{46}$$

where the sums is implied, and $\overleftrightarrow{\mathbb{M}}_i$ are the matrices know as the infinitesimal rotation generators:

$$\overleftrightarrow{\mathbb{M}}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \overleftrightarrow{\mathbb{M}}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \overleftrightarrow{\mathbb{M}}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{47}$$

which have the following propierty:

$$\overleftrightarrow{\mathbb{M}}_i \overleftrightarrow{\mathbb{M}}_j - \overleftrightarrow{\mathbb{M}}_j \overleftrightarrow{\mathbb{M}}_i \equiv [\overleftrightarrow{\mathbb{M}}_i, \overleftrightarrow{\mathbb{M}}_j] = \varepsilon_{ijk} \overleftrightarrow{\mathbb{M}}_k. \tag{48}$$

1.2.3 Rate of change of a vector

By using infinitesimal rotations, we will be able to describe the motion of a rigid body in time. By defining a vector $\vec{g}$, then we can define that the change of $\vec{g}$ as seen by the observer is equal to the change of $\vec{g}$ as seen by the body plus the rotation of the body.

$$(\vec{g})_{space} = (\vec{g})_{body} + (\vec{g})_{rot}, \tag{49}$$

where $(\vec{g})_{rot}$ is given by

$$(\vec{g})_{rot} = d\vec{\Omega} \times \vec{g}, \tag{50}$$

such that the time rate of Eq. (49) can be written as:

$$\left(\frac{d\vec{g}}{dt}\right)_{space} = \left(\frac{d\vec{g}}{dt}\right)_{body} + \vec{\omega} \times \vec{g}, \tag{51}$$

since $d\vec{\Omega} = \vec{\omega}dt$, and $\vec{\omega}$ is the angular velocity of the body.

Hence, it will be convinient to write $\vec{\omega}$ in terms of the temporal derivatives of the Euler angles. By looking back to the transformation found on Eq. (18), we notice that when going from the coordinate space to the primed system (body), a change in ψ around z' affects ω'_z, a change in θ around x' affects ω'_x and ω'_y, and finally a change by ϕ around z, affects all three ω'_x, ω'_y and ω'_z. Therefore, we can express the components of $\vec{\omega}$ with respect to the primed axes as:

$$\begin{aligned}
\omega'_x &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \\
\omega'_y &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \\
\omega'_z &= \dot{\phi}\cos\theta + \dot{\psi}.
\end{aligned} \tag{52}$$

1.2.4 Coriolis Effect

An important problem regarding the kinematics of a rigid body, is the description of the motion of a body relative to the coordinate axes rotating with Earth. Hence, we start by rewriting Eq. (51) as:

$$\vec{v}_s = \vec{v}_b + \vec{\omega} \times \vec{g}, \tag{53}$$

where $\vec{\omega}$ is the constant angular velocity of the Earth relative to the inertial system. By obtaining the time change on Eq. (53):

$$\begin{aligned}
\left(\frac{d^2\vec{r}}{dt^2}\right)_s &= \frac{d}{dt}\left(\left(\frac{d\vec{r}}{dt}\right)_b + \vec{\omega} \times \vec{r}\right) + \vec{\omega} \times \left(\left(\frac{d\vec{r}}{dt}\right)_b + \vec{\omega} \times \vec{r}\right), \\
&= \left(\frac{d^2\vec{r}}{dt^2}\right)_b + 2\left(\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_b\right) + (\vec{\omega}(\vec{\omega} \cdot \vec{r}) - \omega^2\vec{r}),
\end{aligned} \tag{54}$$

where we can identify that the last two terms in Eq. (55) are the coriolis acceleration and the centrifugal acceleration. Hence, we can write the acceleration from the body as

$$\vec{a}_b = \vec{a}_s - 2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_b + ((\omega^2\vec{r}) - \vec{\omega}(\vec{\omega} \cdot \vec{r})). \tag{55}$$

The order of magnitude for a particle on Earth with angular velocity

$$\omega = \frac{2\pi}{24 * 3600} \approx 7 \times 10^{-5} sec^{-1}, \tag{56}$$

then taking the centrifugal acceleration at the equator

$$\vec{a} = \omega^2 \vec{r} - \vec{\omega}(\vec{\omega} \cdot \vec{r})) \approx 3 \times 10^{-2} \frac{m}{s^2}, \tag{57}$$

that is $\approx 3 \times 10^-3g$ or equivalent 0.3% of the acceleration of gravity. This acceleration, even though is small, is not negligible, and there are instances where it becomes of importance.

Now, the Coriolis effect on a moving particular, is perpendicular to both ω and $\frac{d\vec{r}}{dt}$. In the Northern Hemisphere, the Coriolis acceleration $-2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_b$ tends to reflect the projectile along the surface of Earth to the right, while on the Southern Hemisphere, it reflects the opposite direction. By consider a simplified model, when can observe the effects of this phenomena on the circulation of the winds. We first recall that the wind masses tend to move from high-pressure regions to low-pressure regions. When the wind impacts a low-pressure area, the wind will be reflected its right (On the Northern hemisphere). Hence, when several wind fronts impact from several directions a low-pressure area, this tends to form a circular pattern.

Similarly, we can describe quantitatively the effect of the Coriolis acceleration on the launching of projectiles. Assuming that a projectile has a velocity of $10^3 m/s$, the centrifugal acceleration then becomes $a_c = -2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_b \approx 10^{-2} m/s^2$. Thus, for a 2-minute flight ($\approx 10^2 s$), we can calculate by how much the projectile is going to be reflected:

$$\Delta x = \frac{1}{2} a t^2 \approx 10^2 m. \tag{58}$$

Notice that this effect becames larger as the time flight increases.