Classical Mechanics Lecture Notes

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1 Lecture 3/4

1.1 Equations of Motion for Rigid Bodies

For most problems we can write the kinetic energy as:

$$T = T_{cm} + T_{rel} \tag{1}$$

And similarly with the potential energy:

$$V = V_{cm} + V_{rel} \tag{2}$$

Where the subscript cm refers to center of mass part, and rel refers to the relative part, as usual. We also have the center of mass coordinates (X,Y,Z) and we can also have Euler coordinates for the rotation (ϕ, θ, φ)

We can write our angular momentum as:

$$\vec{L} = \sum_i m_i(\vec{r_i} \times \vec{v_i}) \tag{3}$$

Let's rewrite this angular momentum using $\vec{\omega}$

$$\vec{L} = \sum_i m_i[\vec{r_i} \times (\vec{\omega} \cdot \vec{r_i})] \tag{4}$$

If we use the vector identity $A \times (B \times C) = B(A \cdot C) + C(A \cdot B)$ we can write equation (4) as:

$$\vec{L} = m_i(\vec{\omega}\vec{r_i}^2 - \vec{r_i}(\vec{\omega} \cdot \vec{r_i})) \tag{5}$$

And here in equation 5 we are implying summation.
Now lets look at the x-component of the angular momentum. We can write it as:

$$L_x = m_i[\omega_x r_i^2 - \omega_x x_i^2 - \omega_y x_i y_i - \omega_z x_i z_i] \tag{6}$$

Simplifying, we can rewrite this as:

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \tag{7}$$

And, we can use this to define the tensor I_{ij}, and this allows us to write:

$$\vec{L} = \overleftrightarrow{I}\,\vec{\omega} \tag{8}$$

Some interesting points about this tensor I:
-It Operates on $\vec{\omega}$
-It has dimension (versus the dimensionless rotation matrix we were looking at before)
-In general, the determinant $|I| \neq 1$
Now we want to look more closely at the actual elements of this matrix I. We will start first with the diagonal elements:

$$I_{xx} = m_i(r_i^2 - x_i^2)$$

Then the off diagonals:

$$I_{xy} = -m_i x_i y_i$$

We call the diagonal elements the "Moment of Inertia Coefficients" and the off-diagonal elements the "products of inertia".
When we have a continuous system, we simply integrate instead of sum :

$$I_{xx} = \int dm(r^2 - x^2) = \int \rho(\vec{r})(r^2 - x^2)d^3r \tag{9}$$

$$I_{jk} = \int \rho(\vec{r})(r^2\delta_{jk} - x_j x_k)d^3r \tag{10}$$

(There is a potential source of confusion here: This is a slightly different notation from Goldstein's textbook)

1.2 Math Note: Tensors

Before we continue with the talk on the Inertia Tensor, here is a brief section on tensors and vectors. We start with the different types of tensors:
Rank 1 Tensor: This is a vector (at least for our purposes)
Rank 2 Tensor: This is a matrix (again, at least for our purposes)
A tensor T transforms in the following way:

$$\overleftrightarrow{T}' = \overleftrightarrow{A}\overleftrightarrow{T}\overleftrightarrow{A}^{-1} = \overleftrightarrow{A}\overleftrightarrow{T}\overleftrightarrow{A}^{T} \tag{11}$$

(Here we are using A as a rotation matrix) Now we can write T' as:

$$T'_{ij} = \sum_{k,l} a_{ik}T_{kl}a^T_{lj} = \sum_{k,l} a_{ik}T_{kl}a_{jl} \tag{12}$$

Now if we have two vectors $\vec{f}, \vec{g}$ we know that the dot product of these two vectors gives us a scalar:

$$\vec{f} \cdot \vec{g} = scalar$$

We can write these vectors as column vectors of the form:

$$\vec{f} = \begin{pmatrix} f_i \\ . \\ . \\ . \\ f_n \end{pmatrix} \tag{13}$$

And in doing this, we can define our rank 2 tensor using 2 vectors:

$$\overleftrightarrow{T} = \vec{f}\vec{g}^T = \begin{pmatrix} f_i \\ . \\ . \\ . \\ f_n \end{pmatrix} \begin{pmatrix} g_1 & ... & g_n \end{pmatrix} = \begin{pmatrix} f_1 g_1 & ... & f_1 g_n \\ . & . & . \\ . & . & . \\ f_n g_1 & ... & f_n g_n \end{pmatrix} \tag{14}$$

We now wish to see if this matrix we constructed transforms like a tensor. We already know that f and g transform as vectors, so lets look at T. If we look at 3-Dimensions, we can write:

$$T'_{xy} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{xi} T_{ij} a_{yj} = a_{xi} f_i a_{yj} g_j = f'_x g'_y \tag{15}$$

And this shows us that T transforms like a tensor.
Now that we have established these ideas, we can go back to the Inertia tensor.

1.3 The Inertia Tensor

Again we will start with the kinetic energy:

$$T = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m \vec{v_i} \cdot (\vec{\omega} \times \vec{r_i}) \tag{16}$$

Now if we use the vector relation $A \cdot (B \times C) = B \cdot (C \times A)$ we can rewrite this as:

$$T = \frac{\vec{\omega}}{2} \cdot m_i (\vec{r_i} \times \vec{v_i}) = \frac{\vec{\omega} \cdot \vec{L}}{2} = \frac{\vec{\omega}^T \overleftrightarrow{I} \vec{\omega}}{2} \tag{17}$$

We can then write, if we let $\vec{\omega} = \omega \hat{n}$:

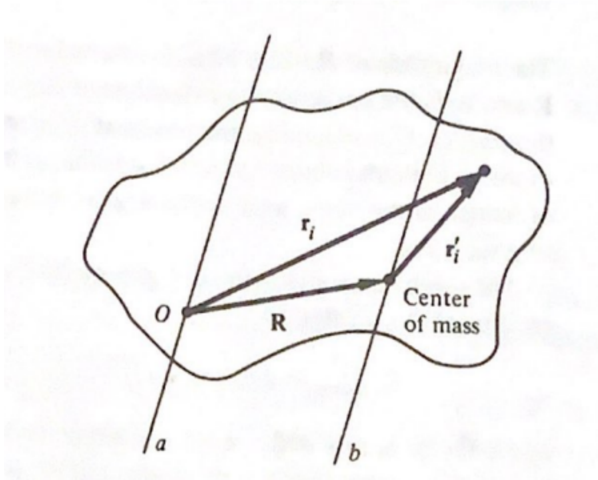
$$T = \frac{\omega^2}{2} \hat{n}^T \overleftrightarrow{I} \hat{n} = \frac{1}{2} I \omega^2 \tag{18}$$

Where here we define $I \equiv \hat{n}^T \overleftrightarrow{I} \hat{n}$, and this is a scalar quantity, so we know that it does not change under a rotation. Finally, we can write the following:

$$I = m_i (\vec{r_i}^2 - (\vec{r_i} \cdot \hat{n}))^2 \tag{19}$$

And you might recall this is actually the definition we would find in a introductory physics course, when we look from the point of view of $\hat{n}$ (the rotation).
We will return to the Inertia tensor in a moment, but first let's discuss the Parallel Axis Theorem.

1.4 Parallel Axis Theorem



If we look at this image from *Goldstein figure 5.4*, we can see how we have a vector $\vec{r_i}$ and a center of mass. We can write the vector $\vec{r_i} = \vec{R} + \vec{r_i}'$. So the moment of Inertia around an axis a is given by

$$I_a = m_i(\vec{r_i} \times \hat{n})^2 = m_i[(\vec{r_i}' + R) \times \hat{n}]^2 \tag{20}$$

Expanding this expression using M as the total mass we can say:

$$I_a = M(\vec{R} \times \hat{n})^2 + m_i(\vec{r_i}' \times \hat{n})^2 + 2(m_i(\vec{R} \times \hat{n}) \cdot (\vec{r_i}' \times \hat{n})) \tag{21}$$

We can actually say that the last term in the above expression is zero, if we write it as $2(\vec{R} \times \hat{n}) \cdot (\Sigma m \vec{r_i}' \times \hat{n})$, noting that this is zero because we are looking at 2 parallel axes with one containing the center of mass.

So we can rewrite this whole expression as

$$I_a = I - b + M(\vec{R} \times \hat{n})^2 = I_b + MR^2 \sin^2\theta \tag{22}$$

where in this expression we note that $MR^2 sin^2\theta$ is the perpendicular displacement of the axis.

1.5 Return to the Inertia Tensor

Recall that we have written

$$T = \frac{1}{2} m_i (\vec{\omega} \times \vec{r_i})^2 \tag{23}$$

We can expand this and write:

$$T = \frac{1}{2} \omega_\alpha \omega_\beta m_i (\delta_{\alpha\beta} r_i^2 - r_{i\alpha} r_{i\beta}) \tag{24}$$

where i is the particle number and α, β are the indices (x,y,z). Then we can write

$$T = \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta \tag{25}$$

Then we can say that an individual element of the Inertia tensor is given by:

$$I_{\alpha\beta} = \int (\delta_{\alpha\beta} r^2 - r_\alpha r_\beta) dm \tag{26}$$

where here dm is the usual mass element. Now we are ready to try a few examples!

1.5.1 Example 1: Homogeneous Cube of side a

Consider a cube of side length a. We wish to find the Inertia tensor for this.

If we start with the diagonal elements we have:

$$I_{xx} = \rho \int_0^a \int_0^a \int_0^a (r^2 - x^2) dx dy dz \tag{27}$$

This isn't a particularly interesting integral, so well skip to the results.

$$I_{xx} = \rho a \left(\frac{zy^3}{3} + \frac{yz^3}{a}\right)\Big|_0^a = \frac{2}{3}\rho a^5 = \frac{2}{3} m a^2 \tag{28}$$

Similarly with the other elements:

$$I_{xy} = -\int_0^a \int_0^a \int_0^a \rho(xy) dx dy dz \tag{29}$$

If we evaluate this we end up with

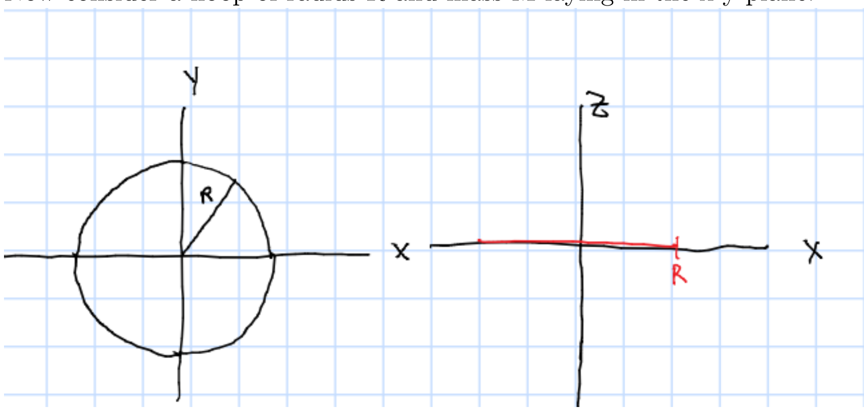
$$I_{xy} = -\frac{1}{4} a^5 \rho = -\frac{1}{4} a^2 m \tag{30}$$

So now we actually have all we need to form our Inertia tensor for this problem!

$$\overleftrightarrow{I} = ma^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} \tag{31}$$

1.5.2 Example 2.1: Hoop of radius R and mass M

Now consider a hoop of radius R and mass M laying in the x-y plane.



Let's also define $\lambda = \frac{M}{2\pi R}$. The individual elements of the Inertia tensor are given by:

$$I_{ab} = \int (r^2\delta_{ab} - r_a r_b)dm = \lambda \int (r^2\delta_{ab} - r_a r_b)Rd\theta \tag{32}$$

Now we can find the elements of this tensor.

$$I_{ab} = \int (r^2\delta_{ab} - r_a r_b)dm \tag{33}$$

Here $dm = \lambda Rd\theta$, So:

$$I_{ab} = \lambda \int (r^2\delta_{ab} - r_a r_b)Rd\theta \tag{34}$$

where:

$$r_x = Rcos\theta, r_y = Rsin\theta, r_z = 0$$

Now we have a few integrals we have to compute to find these elements.

$$I_{xx} = \lambda \int_0^{2\pi} (R^2 - R^2cos^2\theta)Rd\theta = \lambda \int_0^{2\pi} R^3sin^2\theta d\theta = \lambda R^3 \int_0^{2\pi} \frac{1}{2}(1 - cos(2\theta))d\theta = \frac{1}{2}2\pi\lambda R^3 = \frac{1}{2}MR^2 \tag{35}$$

Next

$$I_{yy} = \lambda \int_0^{2\pi} (R^2 - R^2cos^2\theta)Rd\theta = \frac{1}{2}MR^2 \tag{36}$$

$$I_{zz} = \lambda \int_0^{2\pi} (r^2 - 0)Rd\theta = MR^2 \tag{37}$$

Finally

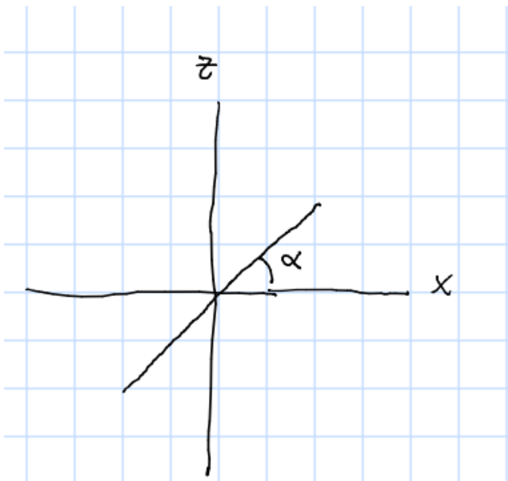
$$I_{xy} = I_{yx} = \lambda \int_0^{2\pi} (R^2cos\theta sin\theta R)d\theta = 0 \tag{38}$$

This allows us to write our tensor as:

$$\overleftrightarrow{I} = MR^2 \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \tag{39}$$

For our next example we will look at what happens if we tilt the hoop.

1.5.3 Example 2.2: Tilted Hoop



We have the same hoop as in the previous example, except now its rotated towards the z axis at an angle α. We know define:
$x = Rcos\theta cos\alpha$
$y = Rsin\theta$
$z = -Rsin\alpha cos\alpha$ We have to do the following integrals to determine our elements now:

$$I_{xx} = \lambda \int_0^{2\pi} (R^2 - R^2 cos^2\theta cos^2\alpha) R d\theta = 2\pi\lambda R^3 - \int_0^{2\pi} = 2\pi\lambda R^3 - \int_0^{2\pi} R^3 cos^2\alpha \frac{1}{2}(1 + cos2\theta) d\theta \tag{40}$$

Computing this integral gives us that

$$I_{xx} = \frac{1}{2} MR^2 (2 - cos^2\alpha) \tag{41}$$

From here, I am going to skip some steps for the integrals to save some paper. The remaining terms are:

$$I_{yy} = \frac{1}{2} MR^2 \tag{42}$$

$$I_{zz} = \lambda \int_0^{2pi} (R^2(1 - \cos^2\theta \sin^2\alpha)) R d\theta = \frac{1}{2} MR^2 (2 - \sin^2\alpha) \tag{43}$$

$$I_{yz} \propto I_{xy} \propto \int sin\theta cos\theta d\theta = 0 \tag{44}$$

$$I_{xz} = \lambda \int_0^{2pi} -R^2 \cos^2\theta \sin\alpha cos\alpha R d\theta = -\lambda R^2 \sin\alpha\cos\alpha \int_0^{2\pi} \frac{1}{2}(1+\cos 2\theta) d\theta = \frac{1}{2} M R^2 \sin\alpha\cos\alpha \tag{45}$$

So we can write the Inertia tensor as:

$$\overleftrightarrow{I} = \frac{1}{2} M R^2 \begin{pmatrix} 2 - cos^2\alpha & 0 & -\frac{1}{2} sin 2\alpha \\ 0 & 1 & 0 \\ -\frac{1}{2} sin\alpha & 0 & 2 - sin^2\alpha \end{pmatrix} \tag{46}$$

and we can rotate our coordinate system by α around y to make I diagonal, as expected.

1.6 Eigenvalues of I and Principal Axes

We know that I is symmetric: $I_{ij} = I_{ji}$ so we have 6 degrees of freedom. We also know that I depends on the origin and axis of rotation. There also exists some combination of origin and axis such that I is diagonal.
In general we can write:

$$\vec{L}^T = (I_1\omega_1, I_2\omega_2, I_3\omega_3) \tag{47}$$

And furthermore, if I is diagonal we can write the kinetic energy as:

$$T = \frac{\vec{\omega}^T \overleftrightarrow{I} \vec{\omega}}{2} = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{48}$$

Now, consider an axis through the center of mass. We can choose a rotation R such that

$$I_D = \overleftrightarrow{R}\overleftrightarrow{I}\overleftrightarrow{R}^T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{49}$$

Then the eigenvectors will lie along $\hat{x'}, \hat{y'}, \hat{y'}$ as defined by I_D. This is called the Principal moment of inertia tensor. The directions of $\hat{x'}, \hat{y'}, \hat{y'}$ are called the principal axes. We can then find any other intertia tensor $\overleftrightarrow{I}$ through the center of mass using an Euler Angle rotation:

$$I = \overleftrightarrow{S}\overleftrightarrow{I_D}\overleftrightarrow{S}^T \tag{50}$$

Then, using the parallel axis theroem, we can translate.
Next we want to know what the eigenvectors are. We can find them using the determinant:

$$\begin{vmatrix} I_{xx} - I_i & I_{xy} & I_{zx} \\ I_{xy} & I_{yy} - I_i & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} - I_i \end{vmatrix} = 0 \tag{51}$$

The eigenvectors are given by:

$$(\overleftrightarrow{I} - I_i \mathbb{1})\vec{a_i} = 0 \tag{52}$$

And example of this are solids of rotation.
Note that the principal moments of inertia cannot be negative, so it follows that

$$I_{xx} = m_i(y_i^2 + z_i^2) \geq 0 \tag{53}$$

If we then the direction of the cosines of the axes to be α, β, γ we can write:

$$\hat{n} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k} \tag{54}$$

We can now write I as:

$$I = \hat{n}^T \overleftrightarrow{I} \hat{n} = I_{xx}\alpha^2 + I_{yy}\beta^2 + I_{zz}\gamma^2 + 2I_{xy}\alpha\beta + 2I_{yz}\beta\gamma + 2I_{zx}\gamma\alpha \tag{55}$$

If we define $\rho = \frac{\hat{n}}{\sqrt{I}}$ we can then write this as:

$$1 = I_{xx}\rho_1^2 + I_{yy}\rho_2^2 + I_{zz}\rho_3^2 + 2I_{xy}\rho_1\rho_2 + 2I_{yz}\rho_2\rho_3 + I_{zx}\rho_3\rho_1 \tag{56}$$

We can always rotate this so that we can have this equation in the form:

$$1 = I_1\rho_1'^2 + I_2\rho_2'^2 + I_3\rho_3'^2 \tag{57}$$

And this equation describes a 3-D elliptical surface. If we want to find the radious of the gyration, we have that

$$R_0^2 = \frac{I}{M} \tag{58}$$

And we can werite the vector $\vec{\rho}$ as

$$\vec{\rho} = \frac{\hat{n}}{R_0\sqrt{M}} \tag{59}$$

2 Lecture 3/9

Now we can finally start solving some problems using some of the methods we developed in the last lecture.

2.1 Solving Rigid Body Problems

First a few general comments:
-If we have a non-holonomic constraint, we have to use Lagrange Multipliers. An example of this is an object rolling.
-If we have a holonomic constriaint, we know that the constraints depend on the positions $f(\{q_i\}) = 0$. This means that we need to choose a set of independent $\{q_i\}$.
-If the axis of rotation is fixed, angles are easy.
-If axis is not fixed, we can either use fixed points (use that point for the origin) or no fixed point, where we would use the center of mass as the origin.

-Use principal axes for coordinate system in the body system.
Now recall that

$$\frac{d\vec{L}}{dt}_{space} = \frac{d\vec{L}}{dt}_{body} + \vec{\omega} \times \vec{L} = \vec{N} \tag{60}$$

We also know that:

$$L_j = I_j \omega_j \tag{61}$$

Here there is no summation, and this is also not a dot product.
From here on, we are mostly only going to work in the body system. So now we can write:

$$\frac{dL_i}{dt} = \epsilon_{ijk} \omega_j L_k = N_i \tag{62}$$

Now we can plug in L, and get 3 equations.

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1 \tag{63}$$

$$I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = N_2 \tag{64}$$

$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3 \tag{65}$$

These 3 equations show us that something cannot spin without torque at a constant $\vec{\omega}$ unless $\vec{\omega}$ is parallel to $\hat{n}_i$, which is any one of the three principal axes. An example of this is tire balancing, where one needs a principal axis to be parallel to the axis of rotation.

2.2 Torque Free Motion

Now lets look at a specific case: Torque Free Motion Here $\vec{N} = 0$, and the object we are looking at is symmetric where $I_1 = I_2$. If we now look at the equations of motion we have:

$$I_3 \dot{\omega}_3 = 0 \tag{66}$$

Since we know $I_1 = I_2$, this allows us to have the following two equations:

$$\dot{\omega_1} + \beta \omega_3 \omega_2 = 0 \tag{67}$$

$$\dot{\omega_2} = \beta \omega_1 \omega_3 = 0 \tag{68}$$

Where we defined $\beta = \frac{I_3 - I_1}{I_1}$. If we solve this system, we get the following

$$\ddot{\omega}_1 + \beta \omega_3 \dot{\omega}_2 = \ddot{\omega}_1 + \beta^2 \omega_3^2 \omega_1 = 0 \tag{69}$$

This gives us

$$\omega_1 = A cos(\beta \omega_3 t + \theta)$$

$$\omega_2 = A sin(\beta \omega_3 t + \theta)$$

And we can define the total angular velocity as:

$$\omega = (\omega_3^2 + A^2)^{\frac{1}{2}} \tag{70}$$

This shows us that we have precession in this example, as well as the fact that the instaneous axis of rotation traces a cone in the body system. This body cone has the 1/2 angle α_b, where

$$tan\alpha_b = \frac{A}{\omega_3}$$

We can then find that

$$A = \omega sin\alpha_b$$

And thus

$$\omega_3 = \omega cos\alpha_b$$

And now we see that hte body is rotating around an axis that is itself rotating. Since $\vec{L}$ is constant, we can also write:

$$cos\alpha_s = \frac{\vec{\omega} \cdot \vec{L}}{\omega L} = \frac{\vec{\omega}^T \overset{\leftrightarrow}{I} \vec{\omega}}{\omega L} = \frac{2T}{\omega L} \tag{71}$$

where the subscript s here referes to the space system, and this equation gives us the space cone, in the space frame. We can determine the shape of these objects, by comparing α_b and α_s:
-If $\alpha_b < \alpha_s$ it is prolate (or cigar-shaped). Here $I_3 > I_1 = I_2$
-If $\alpha_b > \alpha_s$ it is oblate (or disc shaped). Here $I_3 < I_1 = I_2$

2.3 Nondegenerate I

When we have nondegenerate I, we know $I_1 \neq I_2 \neq I_3$.
We also know now that rotation is stable if $\vec{\omega}$ is parallel to $\hat{n_i}$. Let's now think about pertubations.
Let's discuss the case where $\omega_3 >> \omega_1, \omega_2$:
We know that $\dot{\omega_3} \propto \omega_1\omega_2$. This is 2nd order terms that we will neglect it. We can, in the same was as before, solve for ω_1 and ω_2.

$$\omega_1 = A[I_2(I_3 - I_2)]^{1/2} cos(\beta\omega_3 t + \theta) \tag{72}$$

$$\omega_2 = A[I_1(I_3 - I_1)]^{1/2} sin(\beta\omega_3 t + \theta) \tag{73}$$

where we defined

$$\beta = [\frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}]^{1/2}$$

We can discuss three different cases with this solution.
Case 1: $I_3 > I_1, I_2$
Here β is a real number, and the solution is stable.
Case 2: $I_3 < I_1, I_2$
Here β is again a real number, and again we have a stable solution.
Case 3: $I_1 < I_3 < I_2$
This is the interesting case. Here β is complex. The sine and cosine functions

in the solution turn into exponenentials, and the solution becomes unstable. An example of this is a tennis racket, and you can convince yourself of this by thinking about the three rotational axes of the racket, and the stability of them, perhaps by throwing a spinning tennis racket in the air. In that case we should be able to see wich of the three axes the spin is stable.

Let's now look in a more general way. Recall the three equations of motion we got before:

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) \tag{74}$$

$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) \tag{75}$$

$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) \tag{76}$$

Now let's define the following:

$$\vec{\rho} = \frac{\hat{n}}{\sqrt{I}} = \frac{\vec{\omega}}{\omega\sqrt{I}} = \frac{\vec{\omega}}{\sqrt{2T}} \tag{77}$$

$$T = \frac{\vec{\omega}\vec{L}}{2} = \frac{\vec{\omega}^T \overset{\leftrightarrow}{I} \vec{\omega}}{2} = \frac{\omega^2}{2}\hat{n}^T I \hat{n} = \frac{1}{2}I\omega^2 \tag{78}$$

Now we can also define:

$$F(\vec{\rho}) = \vec{\rho}^T \overset{\leftrightarrow}{I} \vec{\rho} = \rho_i^2 I_i \tag{79}$$

Here we again are impliying summation
If $F(\vec{\rho}) = 1$, we have defined an Inertia ellipsoid.
Now, as $\hat{n}$ and $\vec{\omega}$ change, $\vec{\rho}$ is also going to change, however the tip of ρ stays on the inertia ellipsoid. Also, ∇F is normal to the ellipsoid. We can then write:

$$\vec{\nabla}_\rho F = 2\overset{\leftrightarrow}{I}\rho = \sqrt{\frac{2}{T}}\vec{L} \tag{80}$$

And since $\vec{L}$ is constant, $\vec{\nabla}F$ is fixed. $\vec{\omega}$ and $\vec{\rho}$ are constrined to move such that the normal to the ellipsoid points parallel to $\vec{L}$. The ellipsoid then movevs to keep the connection between $\vec{\omega}$ and $\vec{L}$.
The distance from the origin of the ellipsoid and the plane tangent to the ellipsoid at ρ:

$$\frac{\vec{\rho}\cdot\vec{L}}{L} = \frac{\vec{\omega}\cdot\vec{L}}{L\sqrt{2\pi}} = \frac{\sqrt{2T}}{L} = constant \tag{81}$$

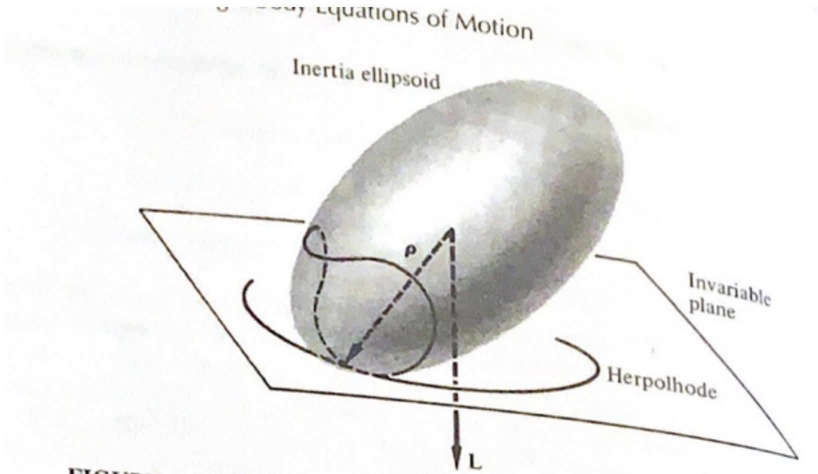


FIGURE 5.4 The motion of the inertia ellipsoid relative to the invariable pl

Goldstein has a nice image of this. In the image we can see that L has a fixed direction, tangent to the invariable plane. The ellipsoid rolls without slipping on the invariable plane as well. Some termonology from the image:
-Polhode is the curve traced out by the point of contact on the ellipsoid
-Herpolhode is the curve on the invariable plane.
We also know that the direction of the angular velocity ω is given by the direction of $\vec{\rho}$, and the orientation of the ellipsoid gives the orientation of the body.
If we go back to the special case where $I_1 = I_2$, we call the inertia ellipsoid the ellipsoid of rotation. The Polehode is thus a circle around the symmetry axis. We then know that $\vec{\omega}$ precesses around the axis of symmetry as well.
Now, if we look generally, and we want to describe the mostion of $\vec{L}$ with respect to the body, we start with the fact that:

$$T = \sum \frac{1}{2}\frac{L_i^2}{I_i} = constant \tag{82}$$

This tells us that L is an ellipsoid that has the equation:

$$\frac{L_1^2}{2TI_1} + \frac{L_2^2}{2TI_2} + \frac{L_3^2}{2TI_3} = 1 \tag{83}$$

Conseration of angular momentum, L, gives us that:

$$L_1^2 + L_2^2 + L_3^2 = L^2 \tag{84}$$

L must therefore lie on the sphere-ellipsoid intersection, meaning:

$$\sqrt{2TI_3} \leq L \leq \sqrt{2TI_1} \tag{85}$$

If we consider stability of motion:

$$L_x : L^2 = 2TI_1 - \epsilon \tag{86}$$

This gives closed figures around L_x.

$$L_z : L^2 = 2TI_3 + \epsilon \tag{87}$$

This gives us closed figures around L_z.
Finally we have no closed figures around L_y, meaning there is unstable motion.
Goldstein Figure 5.5 show this well:

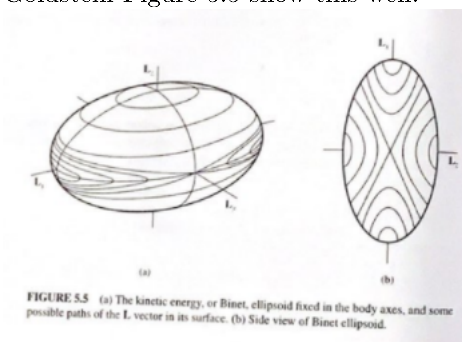


FIGURE 5.5 (a) The kinetic energy, or Binet, ellipsoid fixed in the body axes, and some possible paths of the L vector in its surface. (b) Side view of Binet ellipsoid.

Finally if we go back to the symmetric case $I_1 = I_2$, we can have the following solutions for ω_1 and ω_2

$$\omega_1 = Acos(\beta\omega_3 t + \theta) \tag{88}$$

$$\omega_2 = Asin(\beta\omega_3 t + \theta) \tag{89}$$

Where $\beta = \frac{I_3 - I_1}{I_1}$.
A nice example of this is the precession of the Earth due to the Tidal bulge. This bulge is about 30 km at the equator, and accounts for about 0.5% of the radius of the Earth. We would then have :

$$\frac{I_3 - I_1}{I_1}\ 3 \times 10^{-3}.$$

If β is small, precession is small. The period is about 300 days, so we would expect precession around the axis with period $\simeq$ 300 days. However, what we actually see is deviations in latitude $\simeq 10m$. There is also annual variation (seaonal) that can be attributed to the heating of air.
We also have a 420 day period, which could be due to this precession. However, this would mean that the earth is not rigid, which would imply the existance of a fluid core, which is not unexpected.