3/23 Lecture

Canonical Transformations

Canonical transformations are used to change variables of the Hamiltonian. One of the reasons for changing variables is that the Hamilontian solutions are trivial if it is cyclic in all coordinates q_i. In that case we have

$$p_i = \alpha_i \tag{1}$$

where α_i are constants for all i and then the Hamiltonian H depends on α_i:

$$H = H(\{\alpha_i\}) \tag{2}$$

and

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i \tag{3}$$

where ω_i is a constant. Therefore $q_i = \omega_i t + \beta_i$

In physics, we want to choose coordinates that are the most convenient. For example, if we have a potential that depends on r, then we use polar coordinates instead of Cartesian coordinates.

We have changed variables before.

$$Q_i = Q_i(\{q_j, t\}) \tag{4}$$

which is called a point transformation in configuration space. The transformation from polar to Cartesian coordinates is a point transformation in the configuration space.

Now, for Hamiltonians, we have

$$Q_i = Q_i(\{q_j\}, \{p_j\}, t) \tag{5}$$

$$P_i = P_i(\{q_j\}, \{p_j\}, t) \tag{6}$$

which are still point transformations, but now we are in phase space, not configuration space.

For these to be useful, we require there exists some function $K(\{Q_i\}, \{P_i\}, t)$ such that

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i, \frac{\partial K}{\partial Q_i} = -\dot{P}_i \tag{7}$$

which are Hamilton's equations. Therefore we call K the transformed Hamiltonian. Since this is a general transformation, we assume it to be independent of the problem we are asked. In other words, it should be useful in a Kepler problem as well as in a harmonic oscillator problem.

Just as H, K minimizes the action so we can write

$$\delta \int_{t_1}^{t_2} (P_i\dot{Q}_i - K(\{Q_i\},\{P_i\},t)\,dt \; = 0 \tag{8}$$

$$\delta \int_{t_1}^{t_2} (p_i\dot{q}_i - H(\{q_j\},\{p_j\},t)\,dt \; = 0 \tag{9}$$

which is the small δ variation where the end points and end times are fixed. This gives us

$$\lambda(p_i\dot{q}_i - H) = ((P_i\dot{Q}_i - K) + \frac{\partial F}{\partial t} \tag{10}$$

where the last term will not affect the action integrals given in equations (8) and (9) because the values are the end points are the same. Here, λ is a scale transform. For example,

$$Q_i = \mu q_i, P_i = \nu p_i \tag{11}$$

$$\mu\nu(p\dot{q}_i - H) = P_i\dot{Q}_i - K \tag{12}$$

And we see that the transform only makes values larger or smaller, but doesn't change any of the physics. Since scale transforms are trivial, we will focus on Canonical transformations where $\lambda = 1$. So the transformations change the variables, not the scale. If Q_i and P_i do not depend on t, it is a restricted canonical transformation.

We find that the $\frac{\partial F}{\partial t}$ does not contribute to this action,

$$\delta \int_{t_1}^{t_2} \frac{\partial F}{\partial t} \; = 0 \tag{13}$$

if $F = F(q,p,Q,P,t)$ because

$$\delta q_i = \delta p_j = \delta Q_k = \delta P_l = 0 \tag{14}$$

at the end points. In other words, since the variables q, p, Q, and P are not allowed to change value at the endpoints and F depends only on those variables, F cannot change values at the endpoints and the integral is 0.

Why do we care about F? Because it is the function generating the transforms.

Consider F = F(q, Q, t). Then,

$$\begin{aligned} p_i\dot{q}_i - H &= P_i\dot{Q}_i - K + \frac{dF}{dt} \\ &= P_i\dot{Q}_i - K + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i \end{aligned} \tag{15}$$

For the coefficients of Q_i,

$$(P_i + \frac{\partial F}{\partial Q_i})\dot{Q}_i = 0 \tag{16}$$

giving us $P_i(q, Q, t)$. For the coefficients of q_i,

$$(p_i + \frac{\partial F}{\partial q_i})\dot{q}_i = 0 \tag{17}$$

Giving us $p_i(q, Q, t)$. And

$$K = H + \frac{\partial F}{\partial t} \tag{18}$$

We invert $\{p_i(q, Q, t)\}$ to get $\{Q_i(q, p, t)\}$. Then we use Q_i to get $P_i = -\frac{\partial F}{\partial q_i}$ and find $P_i(q, p, t)$. Similarly, $q_i(Q, P, t)$ and $p_i(Q, P, t)$. And finally we have K(Q, P, t). So we have used our old variables to find the new Hamiltonian, K, in terms of the new variables Q and P. Here we considered one form of a generating function. There are multiple more and summarized in the following table, which was taken from Goldstein.

TABLE 9.1 Properties of the Four Basic Canonical Transformations

Generating Function	Generating Function Derivatives	Trivial Special Case
$F = F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$ $P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$, $Q_i = p_i$, $P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i}$ $Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$, $Q_i = q_i$, $P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i}$ $P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$, $Q_i = -q_i$, $P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i}$ $Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$, $Q_i = p_i$, $P_i = -q_i$

Examples of Canonical Transformation

1) Consider $F_2 = f_i(\{q_j\}t)P_i$ so that the generating function is

$$F = F_2 - Q_i P_i \tag{19}$$

where

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(\{q_i\}t) \tag{20}$$

This is a point transformation and it tells us that all point transformations are canonical transformations.

2) Consider $F_2 = f_i(\{q_j\}t)P_i + g(\{q_j\}t)$. This will give us the same set of $\{Q_i\}$, but a different $\{P_i\}$ because

$$p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial f_i}{\partial q_j}P_i + \frac{\partial g}{\partial q_j} \tag{21}$$

Now a fair question to ask is how can you get different momentum for the same positions? The answer is that we have different Hamiltonians that give different momentum. The equations of motion will still be the same when worked out.

Changing notation, we can write

$$\vec{p} = \frac{\partial \vec{f}}{\partial \vec{q}} \vec{p} + \frac{\partial g}{\partial \vec{q}} \tag{22}$$

which looks weird because we are taking the derivative with respect to a vector. In two dimensions this looks like

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial g}{\partial q_1} \\ \frac{\partial g}{\partial q_2} \end{bmatrix} \tag{23}$$

Where we can invert to get P_1 and P_2 in terms of $\{p_i\}$

3) $F_1 = q_i Q_i$ so we have

$$p_i = \frac{\partial f_1}{\partial q_i} = Q_i \tag{24}$$

and

$$P_i = \frac{\partial F_1}{\partial Q_i} = -q_i \tag{25}$$

which is the first trivial case in the table.

4) Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{p^2 + m^2 w^2 q^2}{2m} \tag{26}$$

Where we will make a canonical transformation such that $p = f(P)\cos(Q)$ and $q = \frac{f(P)\sin(Q)}{mw}$.

$$K = H = \frac{f^2(P)}{2m}(\cos^2(Q) + \sin^2(Q)) = \frac{f^2(P)}{2m} \tag{27}$$

We need to choose f(P) such that the transformation is canonical. We need to come up with the generating function.

$$F_1 = \frac{mwq^2}{2}\cot(Q) \tag{28}$$

So

$$p = \frac{\partial F_1}{\partial q} = mwq\cot(Q) \tag{29}$$

and

$$p = -\frac{\partial F_1}{\partial Q} = \frac{mwq^2}{2\sin^2(Q)} \tag{30}$$

We can solve for q^2:

$$q^2 = \frac{2P\sin^2(Q)}{mw} \rightarrow q = \sqrt{\frac{2P}{mw}}\sin(Q) \tag{31}$$

Do we want the positive or negative root? We don't care because it just changes the phase. However, we will choose the positive root for this example.

Plugging eq. (32) into eq. (30),

$$p = mw\sqrt{\frac{2P}{mw}}\sin(Q)\cot(Q) = \sqrt{2Pmw}\cos(Q) \tag{32}$$

So, $f(P) = \sqrt{2Pmw}$ and therefore

$$H = \frac{f^2(P)}{2m} = wP \tag{33}$$

This is our Hamiltonian. It is cyclic in Q, therefore P is a constant. We have $P = \frac{E}{w}$ and $\dot{Q} = \frac{\partial H}{\partial P} = w$ so

$$Q = wt + \alpha \tag{34}$$

So

$$q = \sqrt{\frac{2E}{mw^2}}\sin(wt + \alpha) \tag{35}$$

$$p = \sqrt{2mE}\cos(wt + \alpha) \tag{36}$$

The graph of q versus p gives an ellipse, which is what we expect for a harmonic oscillator. The graph of Q versus P is constant at E/w.

3/25 Lecture

Symplectic Approach

We are going to start with time independent canonical transformations. Which means $Q_i = Q_i(q,p)$ and $P_i = P_i(q,p)$. This has the advantage of our Hamiltonian, H, not changing. We can write

$$\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j = \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j} \tag{37}$$

and we can invert this to write $p_j = p_j(Q,P)$ and $q_j = q_j(Q,P)$. Then

$$\frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i} + \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i} \tag{38}$$

We want Hamilton's equations to work, so we want $\dot{Q}_i = \frac{\partial H}{\partial P_i}$ which is true if

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial p_j}{\partial P_i} \tag{39}$$

and

$$\frac{\partial Q_i}{\partial p_j} = -\frac{\partial q_j}{\partial P_i} \tag{40}$$

where the LHS of both equations are functions of q, p and the RHS are functions of Q, P. If Eqs. (4) and (5) are true, we can say we have satisfied the symplectic condition and that transformation is canonical.

We can do the same process with

$$\dot{P}_i = \frac{\partial P_i}{\partial q_j}\dot{q}_j + \frac{\partial P_i}{\partial p_j}\dot{p}_j = \frac{\partial P_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial P_i}{\partial p_j}\frac{\partial H}{\partial q_j} \tag{41}$$

and

$$\frac{\partial H}{\partial Q_i} = \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial Q_i} + \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial Q_i} \tag{42}$$

which requires

$$\frac{\partial P_i}{\partial q_j} = -\frac{\partial p_j}{\partial Q_i} \tag{43}$$

and

$$\frac{\partial P_i}{\partial p_j} = \frac{\partial q_j}{\partial Q_i} \tag{44}$$

where, once again, the LHS of both equations are functions of of q, p and the RHS are functions of Q, P. If Eqs. (7) and (8) are satisfied, then we have satisfied the symplectic condition and the transformation is canonical.

Example

Show that $Q = q\cos(\alpha) - p\sin(\alpha)$ and $P = q\sin(\alpha) - p\cos(\alpha)$ is a canonical transformation.

First, we must invert the equations.

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \tag{45}$$

to get

$$\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} \tag{46}$$

Now let's check the symplectic condition

$$\frac{\partial Q}{\partial q} = \cos(\alpha) = \frac{\partial p}{\partial P} \tag{47}$$

$$\frac{\partial Q}{\partial p} = -\sin(\alpha) = -\frac{\partial q}{\partial P} \tag{48}$$

both of which are true, so the symplectic condition is satisfied and the transformation is canonical.

We can use a more compact notation for this concept. We have

$$\vec{\eta}^T = (q_1, ...q_n; p_1, ..., p_n) \tag{49}$$

$$\mathbf{J} = \begin{bmatrix} \mathbb{0} & \mathbb{1} \\ -\mathbb{1} & \mathbb{0} \end{bmatrix} \tag{50}$$

So the vector $\vec{\eta}^T$ contains all our p's and q's and the elemets of $\mathbf{J}$ are n x n matrices. With this compact notation, we have

$$\dot{\vec{\eta}} = \mathbf{J}\frac{\partial H}{\partial \vec{\eta}} \tag{51}$$

which looks weird and confusing because it looks like we are taking the derivative with respect to a vector. What is really is, however, is the vector of the derivatives.

Unraveling the compact notation, we have $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, which is what we expect and an easy check to make sure our new notation works.

Since we are working with canonical transformations, we want a similar compact notation for Q and P:

$$\vec{\xi}^T = (Q_1, ...Q_n; P_1, ..., P_n) \tag{52}$$

and because this is a canonical transformation, the function of Q, P is connected to the function of q, p:

$$\vec{\xi} = \vec{\xi}(\vec{\eta}) \tag{53}$$

Just as at the start of the lecture, we want to derive the symplectic condition, but now using this compact notation.

$$\dot{\xi}_i = \frac{\partial \xi_i}{\partial \eta_j}\dot{\eta}_j \tag{54}$$

with i,j = 1, ..., 2n. We can rewrite it as

$$\dot{\vec{\xi}} = \mathbf{M}\dot{\vec{\eta}} \tag{55}$$

where $\mathbf{M}_{ij} = \frac{\partial \xi_i}{\partial \eta_j}$

So

$$\dot{\vec{\xi}} = \mathbf{M}\mathbf{J}\frac{\partial H}{\partial \vec{\eta}} \tag{56}$$

The reverse transform is

$$\frac{\partial H}{\partial \eta_i} = \frac{\partial H}{\partial \xi_j}\frac{\partial \xi_j}{\partial \eta_i} \tag{57}$$

then

$$\frac{\partial H}{\partial \vec{\eta}} = \mathbf{M}^T \frac{\partial H}{\partial \vec{\xi}} \tag{58}$$

Plugging into Eq. (20),

$$\dot{\vec{\xi}} = \mathbf{MJM}^T \frac{\partial H}{\partial \vec{\xi}} \tag{59}$$

We want $\dot{\vec{\xi}} = \mathbf{J}\frac{\partial H}{\partial \vec{\xi}}$ which is true if $\mathbf{MJM}^T = \mathbf{J}$, which is our symplectic condition in this compact notation. Multiplying to the right of both sides by $\mathbf{M}^{T^{-1}}$ then sandwiching both sides by **J** and **-J**, we find

$$\mathbf{M}^T\mathbf{JM} = \mathbf{J} \tag{60}$$

which is our second symplectic condition for this compact notation.

Example

Consider $F = F_2(q_1, P_1 + F_1(q_2, Q_2)) = q_1P_1 + q_2Q_2$.

$$\vec{\eta} = \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} \tag{61}$$

$$\vec{\xi} = \begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix} \tag{62}$$

Using Table 9.1 from the previous lecture, $Q_1 = q_1, P_1 = p_1$ and $Q_2 = p_2, P_2 = -q_2$

So, from Eq. (19), we have

$$\begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{p}_2 \\ \dot{p}_1 \\ -\dot{q}_2 \end{bmatrix} \tag{63}$$

Using Hamilton's equations (Eq. (51) and Eq. (59)),

$$\begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{P}_1 \\ \dot{P}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\dot{P}_1 \\ -\dot{P}_2 \\ \dot{Q}_1 \\ \dot{Q}_2 \end{bmatrix} \tag{64}$$

We want to check Eq. (24) holds and make sure this is a valid transformation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \mathbf{J} \tag{65}$$

and it is.

It is noteworthy that this derivation is time independent, as noted at the beginning of this lecture, but the conditions are still valid if the canonical transformation is time dependent. Let's show it with a time dependent canonical transform: $\vec{\xi} = \vec{\xi}(\vec{\eta}, t)$. Is this transformation canonical? First, we break it into two transformations: $\vec{\xi} = \vec{\xi}(\vec{\eta}, t_o)$, which is a time independent transformation and we just proved this is canonical. So we look at $\vec{\xi}(t_o) \to \vec{\xi}(t)$ and ask if it is canonical. If this time transformation is canonical then the general time dependent transformation is also canonical as we already showed the time independent transformation is. To do this, we look at a infinitesimal canonical transformation (ICT): $Q_i = q_i + \delta q_i, P_i = p_i + \delta p_i$ so we have

$$\vec{\xi} = \vec{\eta} + \delta\vec{\eta} \tag{66}$$

and we look at

$$F_2 = q_i P_i + \epsilon G(q, P, t) \tag{67}$$

where $q_i P_i$ is our identity generator. We have

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j + \epsilon \frac{\partial G}{\partial q_j} \tag{68}$$

where the last term is δp_j. Also,

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \epsilon \frac{\partial G}{\partial P_j} = q_j + \frac{\partial G}{\partial p_j} \tag{69}$$

in first order of ϵ. We call G the generating function of the ICT. We have

$$\delta q_j = \epsilon \frac{\partial G}{\partial p_j} \tag{70}$$

so

$$\delta\vec{\eta} = \epsilon \mathbf{J} \frac{\partial G}{\partial \vec{\eta}} \tag{71}$$

We want an ICT where $\vec{\xi}(t_o) \to \vec{\xi}(t + dt)$. If this canonical then $\vec{\xi}(t_o) \to \vec{\xi}(t)$ is also canonical. In other words, is Eq. (35) canonical? Let's check:

$$\mathbf{M} = \frac{\partial \vec{\xi}}{\partial \vec{\eta}} = \mathbb{1} + \frac{\partial \delta\vec{\eta}}{\partial \vec{\eta}} = \mathbb{1} + \epsilon \mathbf{J} \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} \tag{72}$$

where $\frac{\partial^2 G}{\partial \vec{\eta}_i \partial \vec{\eta}_j}$ is a matrix.
So

$$\mathbf{M}^T = \mathbb{1} - \epsilon \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} \mathbf{J} \tag{73}$$

and we want to check

$$\mathbf{MJM}^T = (\mathbb{1} + \epsilon \mathbf{J} \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}}) \mathbf{J} (\mathbb{1} - \epsilon \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}} \mathbf{J}) = \mathbf{J} + \epsilon \mathbf{J} \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}}) \mathbf{J} - \mathbf{J} \epsilon \frac{\partial^2 G}{\partial \vec{\eta} \partial \vec{\eta}}) \mathbf{J} = \mathbf{J} \tag{74}$$

so the infinitesimal transform is canonical and therefore a time-dependence transform is canonical. It is important to note that we did not put any restrictions on G, but because we worked in first order of ϵ in Eq. (38) , it doesn't matter what G is. Therefore any infinitesimal transform like the one we considered satisfies the symplectic condition.

We have two canonical transformation formalisms: Generating Function and Symplectic. We can use either to show that canonical transformations have the four properties that of a mathematical "group":

1. The group has an identity, $\mathbb{1}$
2. If $\mathbf{M}$ is a canonical transformation, so is $\mathbf{M}^{-1}$
3. If $\mathbf{M} = \mathbf{M_1 M_2}$, $\mathbf{M}$ is canonical if $\mathbf{M_1}$ and $\mathbf{M_2}$ are canonical.
4. Associative: $\mathbf{M_1(M_2 M_3) = (M_1 M_2) M_3}$

Reffering back to ICT,

$$\delta q_j = \epsilon \frac{\partial G}{\partial p_j} \tag{75}$$

$$\delta p_j = -\epsilon \frac{\partial G}{\partial q_j} \tag{76}$$

choose that G = H, then

$$\delta q_j = \epsilon \frac{\partial H}{\partial p_j} = \epsilon \dot{q_j} \tag{77}$$

and

$$\delta p_j = -\epsilon \frac{\partial H}{\partial q_j} = \epsilon \dot{p_j} \tag{78}$$

These transform

$$Q_i = q_i + \epsilon \dot{q_i} \tag{79}$$
$$P_i = p_i + \epsilon \dot{p_i} \tag{80}$$

which is a transformation forward in time and matches our expectation, proving that this method is valid.