OLD DOMINION UNIVERSITY

Homework #1 Solutions

Course: *Classical Mechanics (Physics 603), Prof. Weinstein*
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Question 1

Consider a marble on a vinyl record, constrained to follow the groove.

a)

Answer. The marble slides inward from R_2 to R_1 in a spiral, taking N rotations around the record. This means that

$$r = R_2 - (R_2 - R_1)\frac{\phi}{2\pi N} = R_2 - \alpha\phi \tag{1}$$

where $\alpha = (R_2 - R_1)/(2\pi N)$. Using

$$\begin{aligned} x &= r\cos\phi & (2)\\ y &= r\sin\phi & (3)\end{aligned}$$

we get

$$\begin{aligned} x &= (R_2 - \alpha\phi)\cos\phi & (4)\\ y &= (R_2 - \alpha\phi)\sin\phi & (5)\end{aligned}$$

b) taking the derivative,

$$\begin{aligned} \dot{x} &= \dot{\phi}\left\{-\alpha\cos\phi - [R_2 - \alpha\phi]\sin\phi\right\} & (6)\\ \dot{y} &= \dot{\phi}\left\{-\alpha\sin\phi + [R_2 - \alpha\phi]\cos\phi\right\} & (7)\\ & & (8)\end{aligned}$$

and

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) & (9)\\ &= \frac{m}{2}\dot{\phi}^2\{\alpha^2\cos^2\phi + 2\alpha(R_2 - \alpha\phi)\sin\phi\cos\phi + (R_2 - \alpha\phi)^2\sin^2\phi & (10)\\ &\quad +\alpha^2\sin^2\phi - 2\alpha(R_2 - \alpha\phi)\sin\phi\cos\phi + (R_2 - \alpha\phi)^2\cos^2\phi\} & (11)\\ &= \frac{m}{2}\dot{\phi}^2(\alpha^2 + r^2) & (12)\end{aligned}$$

where r is defined in terms of ϕ in Eq. 1.

c) the generalized force

$$Q_\phi = F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \quad (13)$$

$$= F_x[-\alpha \cos\phi - (R_2 - \alpha\phi)\sin\phi] + F_y[-\alpha \sin\phi + (R_2 - \alpha\phi)\cos\phi] \quad (14)$$

d) the general equations of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad (15)$$

which gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(m\dot{\phi}\left[\alpha^2 + (R_2 - \alpha\phi)^2\right]\right) \quad (16)$$

$$= m\ddot{\phi}\left[\alpha^2 + (R_2 - \alpha\phi)^2\right] + m\dot{\phi}^2(-2\alpha(R_2 - \alpha\phi)) \quad (17)$$

and

$$\frac{\partial T}{\partial \phi} = \frac{1}{2}m\dot{\phi}^2[2(R_2 - \alpha\phi)(-\alpha)] \quad (18)$$

and

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = m\ddot{\phi}[\alpha^2 + (R_2 - \alpha\phi)^2] - m\dot{\phi}^2[(R_2 - \alpha\phi)\alpha] = Q_\phi \quad (19)$$

Question 2

Equations of motion in a rotating coordinate system

Answer. **a:** (this part should be trivial.)

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad (20)$$

$$m\ddot{x} = F_x \quad (21)$$

$$m\ddot{y} = F_y \quad (22)$$

b: Now transform to

$$q_1 = x\cos\omega t + y\sin\omega t \quad (23)$$

$$q_2 = -x\sin\omega t + y\cos\omega t \quad (24)$$

We can solve for x and y by cross multiplying the above equations for q_1 and q_2 and adding them together to solve for y and for x:

$$q_1\sin\omega t + q_2\cos\omega t = y \quad (25)$$

$$q_1\cos\omega t - q_2\sin\omega t = x \quad (26)$$

or we can recognize that this is just a rotation by ωt and that the inverse rotation is to rotate by $-\omega t$. Diffentiating, we get:

$$\dot{x} = \dot{q}_1 \cos\omega t - q_1\omega \sin\omega t - \dot{q}_2 \sin\omega t - q_2\omega\cos\omega t \quad (27)$$

$$\dot{y} = \dot{q}_1 \sin\omega t + q_1\omega\cos\omega t + \dot{q}_2\cos\omega t - q_2\omega\sin\omega t \quad (28)$$

$$\dot{x}^2 + \dot{y}^2 = \dot{q}_1^2 + q_1^2\omega^2 + \dot{q}_2^2 + q_2^2\omega^2 + (\dot{q}_1 q_1 - \dot{q}_1 q_1)[\ldots] + (\dot{q}_2 q_2 - \dot{q}_2 q_2)[\ldots] \quad (29)$$

$$+(\dot{q}_1\dot{q}_2 - \dot{q}_1\dot{q}_2)[\ldots] - \omega^2(q_1 q_2 - q_1 q_2)[\ldots] - 2\omega\dot{q}_1 q_2 + 2\omega q_1\dot{q}_2 \quad (30)$$

$$T = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + 2\omega(q_1\dot{q}_2 - \dot{q}_1 q_2) + \omega^2(q_1^2 + q_2^2) \quad (31)$$

where the $[\ldots]$ are irrelevant terms multiplying stuff that cancels. The generalized forces and equations of motion are

$$Q_1 = F_x\cos\omega t + F_y\sin\omega t \quad (32)$$

$$Q_2 = -F_x\sin\omega t + F_y\cos\omega t \quad (33)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \quad (34)$$

giving the equations of motion

$$\frac{d}{dt}(m\dot{q}_1 - m\omega q_2) - \frac{1}{2}m(2\omega\dot{q}_2 + 2\omega^2 q_1) = Q_1 \quad (35)$$

$$\frac{d}{dt}(m\dot{q}_2 + m\omega q_1) - \frac{1}{2}m(-2\omega\dot{q}_1 + 2\omega^2 q_2) = Q_2 \quad (36)$$

which can be rearranged to look like:

$$m\ddot{q}_1 - 2m\omega\dot{q}_2 - m\omega^2 q_1 = F_x\cos\omega t + F_y\sin\omega t \quad (37)$$

$$m\ddot{q}_2 + 2m\omega\dot{q}_1 - m\omega^2 q_2 = -F_x\sin\omega t + F_y\cos\omega t \quad (38)$$

Question 3

Reduced mass

Answer. We start with the definitions:

$$\mathbf{R} = \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{m_1 + m_2} \quad (39)$$

$$\mathbf{V} = \dot{\mathbf{R}} \quad (40)$$

$$M = m_1 + m_2 \quad (41)$$

$$T = \frac{1}{2}m_2 v_1^2 + \frac{1}{2}m_2 v_2^2 \quad (42)$$

We now write T in terms of V and v (the center of mass and relative velocities) and work to recover the equation for T in terms of v_1 and v_2:

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\mu v^2 \quad (43)$$

$$= \frac{1}{2}(m_1 + m_2)\frac{(m_1\mathbf{v_1} + m_2\mathbf{v_2})^2}{(m_1 + m_2)^2} + \frac{1}{2}\mu(\mathbf{v_1} - \mathbf{v_2})^2 \quad (44)$$

$$= \frac{m_1^2 v_1^2 + 2m_1 m_2\mathbf{v_1}\cdot\mathbf{v_2} + m_2^2 v_2^2}{m_1 + m_2} + \frac{1}{2}\mu(v_1^2 - 2\mathbf{v_1}\cdot\mathbf{v_2} + v_2^2) \quad (45)$$

where we need the $\mathbf{v_1} \cdot \mathbf{v_2}$ terms to cancel. This gives the constraint that

$$\frac{m_1 m_2 \mathbf{v_1} \cdot \mathbf{v_2}}{m_1 + m_2} - \mu \mathbf{v_1} \cdot \mathbf{v_2} = 0 \tag{46}$$

or that $\mu = m_1 m_2/(m_1 + m_2)$. We still need to show that T is the same as before. Substituting this back into the expression for T, we have:

$$\begin{aligned} T &= \frac{1}{2}\frac{m_1^2 v_1^2 + m_2^2 v_2^2}{m_1 + m_2} + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(v_1^2 + v_2^2) && (47) \\ &= \frac{1}{2}\frac{m_1^2 v_1^2 + m_1 m_2 v_1^2}{m_1 + m_2} + \frac{1}{2}\frac{m_2^2 v_2^2 + m_1 m_2 v_2^2}{m_1 + m_2} && (48) \\ &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 && (49) \end{aligned}$$

Note that you can also solve this problem starting with Eq. 1.31 $T = \frac{1}{2}MV^2 + \frac{1}{2}m_1(v_1')^2 +$ $+\frac{1}{2}m_2(v_2')^2$ instead.

Question 4

Two masses on a spring after a collision

Answer. From problem 3 we know that $T = \frac{1}{2}MV_{tot}^2 + \frac{1}{2}\mu v^2$ where $\mathbf{V_{tot}} = \dot{\mathbf{R}}$ and $\mathbf{v} = \dot{\mathbf{r}} = (\dot{\mathbf{x}}_\mathbf{2} - \dot{\mathbf{x}}_\mathbf{1})$. The potential energy is

$$V = \frac{1}{2}k(r - L)^2$$

and we have

$$\begin{aligned} m_1' &= m_1 + m && (50) \\ M &= m + m_1 + m_2 && (51) \\ \mu &= \frac{m_1' m_2}{M} && (52) \end{aligned}$$

and

$$\mathcal{L} = T - V$$

giving the equations of motion:

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{R}}\right) - \frac{\partial \mathcal{L}}{\partial R} = M\ddot{R} - 0 && (53) \\ 0 = &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = \mu\ddot{r} + k(r - L) && (54) \end{aligned}$$

which gives $\dot{R}$ equals a constant, which must equal the initial CM (center of mass) velocity:

$$V_{tot}^0 = \frac{m v_0}{M}$$

by conservation of momentum.

We also have

$$\ddot{r} = -\frac{k}{\mu}(r - L)$$

which gives

$$r - L = A\cos\omega t + B\sin\omega t$$

where $\omega = \sqrt{k/\mu}$.

Using the boundary conditions:

1. $r = L$ at $t = 0$ gives $A = 0$
2. $v = (-mv_0)/m_1'$ at $t = 0$ gives $B = -v_0 m/(\omega m_1')$

Question 5

Rocket with a drag force

Answer. **a:** using conservation of momentum

$$\frac{dP}{dt} = \frac{dP_R}{dt} + \frac{dP_F}{dt} \quad (55)$$

$$= (m_R + m_F(t))\dot{v} + \left[-\frac{dm_F(t)}{dt}\right](-u) \quad (56)$$

$$= (m_R + m_F(t))\dot{v} - u\gamma = -Av^2 \quad (57)$$

so that we get the differential equation:

$$\dot{v} = \frac{u\gamma - Av^2}{m_R + m_F(t)}$$

b: when moving at constant speed, $\dot{v} = 0$ and

$$u\gamma = -Av^2$$

c: After the rocket runs out of fuel, $\gamma = 0$ and $m_F = 0$, giving

$$\dot{v} = -\frac{Av^2}{m_R}$$

which can be rearranged to give

$$\frac{dv}{v^2} = -(A/m_R)dt$$

which can be integrated to give

$$v(t) = (at + b)^{-1}$$

where we use the fact that $v = v_0$ at $t = 0$ to give $b = 1/v_0$ and

$$\dot{v} = \frac{-1}{(at+b)^2} = \frac{-Av^2}{m_R}$$

to give $a = A/m_R$.

d: bonus This is the more interesting part. In time dt, the rocket will pass through a volume $\pi R^2 v dt$ and hit a number of particles $dn = N\pi R^2 v dt$.

In the reference frame of the rocket, the particles hit with velocity $\mathbf{v} = -v\hat{x}$ and bounce off isotropically in the $y - z$ plane. This means that the average change in momentum in both $\hat{y}$ and $\hat{z}$ averages to zero, and the change in momentum in the $\hat{x}$ direction is

$$dp = -mvdn = -mN\pi R^2 v^2 dt$$

and

$$\frac{dp}{dt} = -mN\pi R^2 v^2$$

giving a drag force proportional to v^2 with a proportionality constant

$$A = mN\pi R^2$$