OLD DOMINION UNIVERSITY

Homework #2 Solutions

Course: *Classical Mechanics (Physics 603), Prof. Weinstein*
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Question 1

Two particles connected by a string, one on a table and the other hanging down.

Answer. The generalized coordinates are (r, θ) where

$$x_1 = r\cos\theta \quad (1)$$
$$y_1 = r\sin\theta \quad (2)$$
$$z_2 = r - L \quad (3)$$

which gives

$$T = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}\dot{z}_2^2 \quad (4)$$
$$V = m_2 g z_2 = m_2 g(r - L) \quad (5)$$
$$T = \frac{m_1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m_2}{2}\dot{r}^2 \quad (6)$$
$$\mathcal{L} = T - V \quad (7)$$

so that the ELE (Euler-Lagrange Eq) give

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = \frac{1}{2}(m_1 + m_2)\ddot{r} - m_1 r\dot{\theta}^2 + m_2 g = 0 \quad (8)$$

At equilibrium, $\ddot{r} = 0$ and this gives

$$m_1 r\dot{\theta}^2 = m_2 g$$

or that the centripetal force on mass 1 comes from the gravitational force on mass 2.

The 2nd ELE gives:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = m_1 r^2\ddot{\theta} + 2m_1 r\dot{r}\dot{\theta} = 0$$

At equilibrium, $\ddot{\theta} = 0$ so that

$$\dot{\theta} = \sqrt{\frac{m_2 g}{m_1 r}}$$

is nonzero and therefore $\dot{r} = 0$ also.

Question 2

A point mass on a plane tethered to the origin by a string of length r. Use Lagrange multipliers to solve this.

Answer. We start with

$$T = \frac{m}{2}\dot{x}^2 + \frac{m}{2}\dot{y}^2 \tag{9}$$

$$V = 0 \tag{10}$$

$$g(x,y) = x^2 + y^2 - R^2 = 0 \tag{11}$$

where g is the constraint equation. This gives

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

and the same for y, which gives the eq of motion

$$m\ddot{x} = 2x\lambda \tag{12}$$

$$m\ddot{y} = 2y\lambda \tag{13}$$

Now we use the ansatz that $x = R\cos\phi$ and $y = R\sin\phi$ in order to satisfy the constraint $g = 0$. This gives the familiar

$$\dot{x} = -R\sin\phi\dot{\phi} \tag{14}$$

$$\ddot{x} = -R\cos\phi\dot{\phi}^2 - R\sin\phi\ddot{\phi} \tag{15}$$

$$\ddot{y} = -R\sin\phi\dot{\phi}^2 + R\cos\phi\ddot{\phi} \tag{16}$$

which we plug into the eq of motion to get

$$m(-R\cos\phi\dot{\phi}^2 - R\sin\phi\ddot{\phi}) = 2\lambda R\cos\phi \tag{17}$$

$$m(-R\sin\phi\dot{\phi}^2 + R\cos\phi\ddot{\phi}) = 2\lambda R\sin\phi \tag{18}$$

and then cross multiply by $\sin\phi$ (top eq) and $\cos\phi$ (bottom eq) and subtract to get

$$MR\ddot{\phi} = 0$$

which gives $\dot{\phi} = \omega = constant$ and $\phi = \omega t$. We can then use this to get

$$-mR\dot{\phi}^2\cos\phi = 2\lambda R\cos\phi$$

which allows as us to solve for λ to get

$$\lambda = \frac{1}{2}m\dot{\phi}^2$$

Now we can find the forces of constraint:

$$Q_x = \lambda\frac{\partial f}{\partial x} = -\frac{m\omega^2}{2}2x = -m\omega^2 x \tag{19}$$

$$Q_y = \lambda\frac{\partial f}{\partial x} = -m\omega^2 y \tag{20}$$

Question 3

Two masses m_1 and m_2 move under their mutual gravitational attraction in a uniform external gravitational field whose acceleration is g.

Answer. We start by writing the kinetic and potential energy in cartesian coordinates and convert to polar coordinates for the relative coordinates.

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 \tag{21}$$
$$V = MgZ - \frac{G\mu M}{r} \tag{22}$$
$$\mathcal{L} = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - MgZ + \frac{G\mu M}{r} \tag{23}$$

since $\mu M = m_1 m_2$. Then the generalized momenta are $p_j = \partial\mathcal{L}/\partial\dot{q}_j$ which gives

$$p_X = M\dot{X} \tag{24}$$
$$p_Y = M\dot{Y} \tag{25}$$
$$p_Z = M\dot{Z} \tag{26}$$
$$p_r = \mu\dot{r} \tag{27}$$
$$p_\theta = \mu r^2\dot{\theta} \tag{28}$$
$$p_\phi = \mu r^2\sin^2\theta\dot{\phi} \tag{29}$$

where p_X, p_Y and p_ϕ are conserved because $\mathcal{L}$ does not depend on the corresponding coordinates. The equations of motion are

$$0 = M\ddot{X} = M\ddot{Y} \tag{30}$$
$$M\ddot{Z} = -Mg \tag{31}$$
$$\dot{p}_\theta = 2\mu r\dot{r}\dot{\theta} + \mu r^2\ddot{\theta} = 0 + \mu r^2\sin\theta\cos\theta\dot{\phi}^2 \tag{32}$$
$$\dot{p}_\phi = 2\mu r\dot{r}\sin^2\theta\dot{\phi} + 2\mu r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + \mu r^2\sin^2\theta\ddot{\phi} = 0 \tag{33}$$
$$\dot{p}_r = \mu\ddot{r} = -\frac{G\mu M}{r^2} + \mu(r\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) \tag{34}$$

where Mg is the expected gravitational forces and the $\dot{p}_r$ term has the expected gravitational and centripetal terms.

Question 4

Minimize the action for a falling object with a varied path.

Answer. We will write the Lagrangian in terms of s and $\dot{s}$, substitute in for the specified path, and then integrate to get the action. s is the distance fallen (so larger s is a smaller height).

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{s}^2 + mgs \tag{35}$$
$$s = \frac{1}{2}gt^2 + A\sin(\pi t/T) \tag{36}$$
$$\dot{s} = gt + \frac{A\pi}{T}\cos(\pi t/T) \tag{37}$$
$$\mathcal{L} = \frac{1}{2}m\left(g^2t^2 + \frac{2gA\pi}{T}t\cos(\pi t/T) + \frac{A^2\pi^2}{T^2}\cos^2(\pi t/T)\right) + \tag{38}$$
$$mg(\frac{1}{2}gt^2 + A\sin(\pi t/T)) \tag{39}$$

so now we integrate over time and take the derivative with respect to A and set it to zero (taking the derviative first to simplify the algebra):

$$\frac{d}{dA}\int_0^T \mathcal{L} dt = \int_0^T \left[\frac{mg\pi t}{T}\cos(\pi t/T) + \frac{mA\pi^2}{T^2}\cos^2(\pi t/T) + mg\sin(\pi t/T)\right] dt \quad (40)$$

Let's look at each of the three terms on the right side. First we integrate $t\cos at$ by parts using

$$\int_0^T t\cos(at)dt = \frac{t}{a}\sin(at)|_0^T - \int_0^T \frac{\sin(at)}{a}dt$$

which gives the first term:

$$\int_0^T \frac{mg\pi}{T} t\cos(\pi t/T)dt = \frac{mg\pi}{T}\frac{T}{\pi} t\sin(\pi t/T)|_0^T - \frac{mg\pi}{T}\frac{T^2}{\pi^2}(-\cos(\pi t/T))|_0^T \quad (41)$$

$$= 0 - \frac{2mgT}{\pi} \quad (42)$$

The second term needs the cosine double angle formula $\cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$:

$$\int_0^T \frac{mA\pi^2}{T^2}\frac{1}{2}(1+\cos(2\pi t/T)dt = \frac{1}{2}\frac{mA\pi^2}{T} + 0$$

and the third term can be directly integrated as

$$\int_0^T mg\sin(\pi t/T)dt = -\frac{mgT}{\pi}\cos(\pi t/T)|_0^T = \frac{2mgT}{\pi}$$

Summing the three terms gives

$$\frac{d}{dA}\int_0^T \mathcal{L} dt = \frac{2mgT}{\pi} + \frac{1}{2}\frac{mA\pi^2}{T} - \frac{2mgT}{\pi} = 0$$

Therefore $A = 0$ and the proposed sinusoidal variation to the path does not minimize the action.