Question 1

1D particle with force \( F = -kx + ax^{-3} \) where \( k, a > 0 \). Find equilibria and the period of small oscillations.

**Answer.**

**a:** The equilibria occur at \( F = -\left(\frac{\partial V}{\partial x}\right) = 0 \) so that

\[ -kx + ax^{-3} = 0 \]

which gives

\[ x = \pm \sqrt[4]{\frac{a}{k}} \]

(real roots only).

Checking stability:

\[ \frac{\partial^2 V}{\partial x^2} = -\frac{\partial F}{\partial x} \]

\[ = k + 3ax^{-4} \]

\[ = k + 3\frac{a^4}{k} = 4k > 0 \]  

**b:** small oscillations

\[ V(x) = V(x_0) + \frac{\partial V}{\partial x} \delta x + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} (\delta x)^2 + \ldots \]

\[ = V_0 + \frac{1}{2} (4k)(\delta x)^2 + \ldots \]

This is a harmonic oscillator with frequency \( \omega = \sqrt{4k/m} \) for both minima.

Question 2

Two masses in circular orbit around their common CM with period \( \tau \). Their motion is suddenly stopped and they fall directly toward each other. Prove that they hit at time \( T = \tau / (4\sqrt{2}) \).
Answer. This reduces to one particle with relative motion described by $\mathcal{L} = \frac{1}{2} \mu \dot{\vec{r}}^2 - \frac{G\mu M}{r}$ in a circular orbit of radius $R$. The period of a circular or elliptical orbit is

$$\tau = 2\pi a^{3/2} \sqrt{\frac{M}{k}}$$

where $a$ is the semi major axis.

Let’s look at this in three ways. First, as we slowly reduce the orbital speed of the mass from $r \dot{\theta}$ to 0, the orbit becomes elliptical with $e$ increasing from 0 to 1. As this happens, the aphelion distance stays the same ($R$) and the perihelion distance decreases from $R$ to 0. This means that the semi major axis $a$ decreases from $R$ to $R/2$ as the “diameter” of the orbit decreases from $2R$ to $R$. This will decrease the period $\tau$ by a factor of $2\sqrt{2}$. Since the period is the time it takes for the particle to fall from aphelion to perihelion and back, the collision time will be half the period. Thus

$$T = \frac{\tau}{4\sqrt{2}}.$$

Second, let’s use more equations. The energy in a circular orbit is

$$E = T + V = \frac{1}{2} V < 0.$$ 

When the motion is stopped, then

$$E' = T + V = 0 + V = 2E < 0$$

and the energy is doubled. Using Eq. 3.61,

$$a = -\frac{k}{2E} = R \quad (6)$$
$$a' = -\frac{k}{2E'} = \frac{1}{2} a \quad (7)$$

This confirms our intuition from the first discussion that the semi major axis will decrease by a factor of two.

Third, let’s brute force it. The total energy when stopped is

$$E = -\frac{GM\mu}{R}$$

and energy conservation gives

$$\frac{1}{2} \mu \dot{r}^2 - \frac{GM\mu}{r} = -\frac{GM\mu}{R}$$

so that

$$\dot{r} = -\sqrt{\frac{2GM}{R}} \sqrt{\frac{R-r}{r}}$$

where the minus sign indicates that $r$ is decreasing. Solving for $dt$ and integrating:

$$T = -\sqrt{\frac{2GM}{R}} \int_R^0 \sqrt{\frac{r}{R-r}} dr$$
Using the substitution $r = R \sin^2 \xi$, and reversing the integration limits:

$$
\int_{\theta}^{\pi/2} \sqrt{\frac{R}{R-r}} \, dr = \int_{0}^{\pi/2} \sqrt{\frac{R}{R - R \sin^2 \xi}} \sin \xi R2 \sin \xi \cos \xi d\xi
$$

(8)

$$
= 2R \int_{0}^{\pi/2} \sin^2 \xi d\xi
$$

(9)

$$
= 2R \int_{0}^{\pi/2} (1 - \cos(2\xi))d\xi
$$

(10)

$$
= R[\xi + \frac{1}{2} \sin(2\xi)]|_{0}^{\pi/2}
$$

(11)

$$
= \frac{1}{2} \pi R
$$

(12)

which yields

$$
T = \sqrt{R \frac{\pi R}{2GM}} = 2\pi \sqrt{\frac{R^3}{2GM} \frac{1}{4\sqrt{2}}} = \frac{\tau}{4\sqrt{2}}
$$

Question 3

Find the ratio of the maximum and minimum velocities for a planet circling the Sun with eccentricity $e$.

**Answer.** We know the maximum and minimum distances are $r_{ap} = a(1 + e)$ and $r_{peri} = a(1 - e)$. We also know that $\dot{r} = 0$ at these extrema so the velocity is perpendicular to $\vec{r}$. This means that the angular momentum at both locations is $l = mv\vec{r}$ so that

$$
mv_{ap}r_{ap} = mv_{peri}r_{peri}
$$

which gives

$$
\frac{v_{peri}}{v_{ap}} = \frac{r_{ap}}{r_{peri}} = \frac{1 + e}{1 - e}
$$

and the planet is moving fastest at perihelion where $r$ is minimum. For the Earth orbiting the Sun, this amounts to

$$
\frac{r_{ap}}{r_{peri}} = \frac{1 + e}{1 - e}|_{Earth} = \frac{1 + 0.0167}{1 + 0.0167} = 1.034
$$

and a 3.4% difference in speed is quite noticeable.