OLD DOMINION UNIVERSITY

Homework #3 Solutions

Course: *Classical Mechanics (Physics 603), Prof. Weinstein*
Spring 2021

Question 1

1D particle with force $F = -kx + ax^{-3}$ where $k, a > 0$. Find equilibria and the period of small oscillations.

Answer. **a:** The equilibria occur at $F = -(\partial V/\partial x) = 0$ so that

$$-kx + ax^{-3} = 0$$

which gives

$$x = \pm \sqrt[4]{a/k}$$

(real roots only).

Checking stability:

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\partial F}{\partial x} \tag{1}$$
$$= k + 3ax^{-4} \tag{2}$$
$$= k + 3a\frac{k}{a} = 4k > 0 \tag{3}$$

b: small oscillations

$$V(x) = V(x_0) + \frac{\partial V}{\partial x}\delta x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\delta x)^2 + \dots \tag{4}$$
$$= V_0 + \frac{1}{2}(4k)(\delta x)^2 + ... \tag{5}$$

This is a harmonic oscillator with frequency $\omega = \sqrt{4k/m}$ for both minima.

Question 2

Two masses in circular orbit around their common CM with period τ. Their motion is suddenly stopped and they fall directly toward each other. Prove that they hit at time $T = \tau/(4\sqrt{2})$.

Answer. This reduces to one particle with relative motion described by $\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - \frac{G\mu M}{r}$ in a circular orbit of radisu R. The period of a circular or elliptical orbit is

$$\tau = 2\pi a^{3/2}\sqrt{\frac{M}{k}}$$

where a is the semi major axis.

Let's look at this in three ways. First, as we slowly reduce the orbital speed of the mass from $r\dot{\theta}$ to 0, the orbit becomes elliptical with e increasing from 0 to 1. As this happens, the aphelion distance stays the same (R) and the perihelion distance decreases from R to 0. This means that the semi major axis a decreases from R to $R/2$ as the "diameter" of the orbit decreases from $2R$ to R. This will decrease the period τ by a factor of $2\sqrt{2}$. Since the period is the time it takes for the particle to fall from aphelion to perihelion and back, the collision time will be half the period. Thus

$$T = \frac{\tau}{4\sqrt{2}}.$$

Second, let's use more equations. The energy in a circular orbit is

$$E = T + V = \frac{1}{2}V < 0.$$

When the motion is stopped, then

$$E' = T + V = 0 + V = 2E < 0$$

and the energy is doubled. Using Eq. 3.61,

$$a = -\frac{k}{2E} = R \tag{6}$$

$$a' = -\frac{k}{2E'} = \frac{1}{2}a \tag{7}$$

This confirms our intuition from the first discussion that the semi major axis will decrease by a factor of two.

Third, let's brute force it. The total energy when stopped is

$$E = -\frac{GM\mu}{R}$$

and energy conservation gives

$$\frac{1}{2}\mu\dot{r}^2 - \frac{GM\mu}{r} = -\frac{GM\mu}{R}$$

so that

$$\dot{r} = -\sqrt{\frac{2GM}{R}}\sqrt{\frac{R-r}{r}}$$

where the minus sign indicates that r is decreasing. Solving for dt and integrating:

$$T = -\sqrt{\frac{2GM}{R}}\int_R^0 \sqrt{\frac{r}{R-r}}dr$$

Using the substitution $r = R\sin^2\xi$, and reversing the integration limits:

$$\int_0^R \sqrt{\frac{r}{R-r}}dr = \int_0^{\pi/2} \sqrt{\frac{R}{R - R\sin^2\xi}} \sin\xi R 2 \sin\xi \cos\xi d\xi \quad (8)$$

$$= 2R \int_0^{\pi/2} \sin^2\xi d\xi \quad (9)$$

$$= 2R \int_0^{\pi/2} (1 - \cos(2\xi)) d\xi \quad (10)$$

$$= R[\xi + \frac{1}{2}\sin(2\xi)]|_0^{\pi/2} \quad (11)$$

$$= \frac{1}{2}\pi R \quad (12)$$

which yields

$$T = \sqrt{\frac{R}{2GM}}\frac{\pi R}{2} = 2\pi\sqrt{\frac{R^3}{GM}}\frac{1}{4\sqrt{2}} = \frac{\tau}{4\sqrt{2}}$$

Question 3

Find the ratio of the maximum and minimum velocities for a planet circling the Sun with eccentricity e.

Answer. We know the maximum and minimum distances are $r_{ap} = a(1+e)$ and $r_{peri} = a(1-e)$. We also know that $\dot{r} = 0$ at these extrema so the velocity is perpendicular to $\vec{r}$. This means that the angular momentum at both locations is $l = mvr$ so that

$$mv_{ap}r_{ap} = mv_{peri}r_{peri}$$

which gives

$$\frac{v_{peri}}{v_{ap}} = \frac{r_{ap}}{r_{peri}} = \frac{1+e}{1-e}$$

and the planet is moving fastest at perihelion where r is minimum.

For the Earth orbiting the Sun, this amounts to

$$\frac{r_{ap}}{r_{peri}} = \frac{1+e}{1-e}|_{Earth} = \frac{1+0.0167}{1+0.0167} = 1.034$$

and a 3.4% difference in speed is quite noticeable.