OLD DOMINION UNIVERSITY

Homework #4 Solutions

Course: *Classical Mechanics (Physics 603), Prof. Weinstein*
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Question 1

Goldstein problem 6.4. Two coupled pendula with small oscillations and same length pendula.

Answer. Note that I am defining θ_1 and θ_2 so that both are positive counter-clockwise. The picture in Goldstein defines them opposite to each other which leads to an irritating and unnecessary minus sign.

The position of the upper pendulum bob is

$$\begin{aligned} x_1 &= l\sin\theta_1 \approx l\theta_1 && (1)\\ y_1 &= l(1-\cos\theta_1) \approx \frac{1}{2}l\theta_1^2 && (2)\end{aligned}$$

and the position of the lower pendulum is

$$\begin{aligned} x_2 &= l\sin\theta_1 + l\sin\theta_2 \approx l\theta_1 + l\theta_2 && (3)\\ y_2 &= l(1-\cos\theta_1) + l(1-\cos\theta_2) \approx \frac{1}{2}l\theta_1^2 + \frac{1}{2}l\theta_2^2 && (4)\end{aligned}$$

and the corresponding velocities are

$$\begin{aligned} \dot{x}_1^2 &= l^2\dot{\theta}_1^2 && (5)\\ \dot{y}_1^2 &= l^2\theta_1^2\dot{\theta}_1^2 \to 0 && (6)\\ \dot{x}_2^2 &= l^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) && (7)\\ \dot{y}_2^2 &= 0 && (8)\end{aligned}$$

where the $\dot{y}$ terms are zero because they are 4th order in θ or $\dot{\theta}$. Then

$$T = \frac{1}{2}l^2[m_1\dot{\theta}_1^2 + m_2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)]$$

or

$$\mathbf{T} = l^2\begin{pmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}.$$

Similarly

$$V = \frac{1}{2}m_1 g l\theta_1^2 + \frac{1}{2}m_2 g l(\theta_1^2 + \theta_2^2)$$

or

$$\mathbf{V} = lg\begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix}$$

which lets us write

$$\mathcal{L} = T - V \quad (9)$$
$$= \frac{1}{2}\dot{\vec{\theta}}^T \mathbf{T} \dot{\vec{\theta}} - \frac{1}{2}\vec{\theta}^T \mathbf{V} \vec{\theta} \quad (10)$$

where $\vec{\theta} = (\theta_1, \theta_2)$.

Solving the eigenvalue equation $(\mathbf{V} - \lambda \mathbf{T})\vec{a} = \vec{0}$ means that

$$0 = \begin{vmatrix} (m_1+m_2)(g-\lambda l) & -\lambda l m_2 \\ -\lambda l m_2 & m_2(g-\lambda l) \end{vmatrix}$$

giving the characteristic equation

$$(m_1+m_2)m_2(g^2 - 2gl\lambda + \lambda^2 l^2) - \lambda^2 l^2 m_2^2 = 0$$

$$\lambda^2 + \lambda\frac{-2g}{l}(1+\frac{m_2}{m_1}) + \frac{g^2}{l^2}(1+\frac{m_2}{m_1}) = 0$$

solving for lambda

$$\lambda = \frac{1}{2}\left[\frac{2g}{l}(1+\frac{m_2}{m_1}) \pm \left[\frac{4g^2}{l^2}(1+2\frac{m_2}{m_1}+\frac{m_2^2}{m_1^2}) - \frac{4g^2}{l^2}(1+\frac{m_2}{m_1})\right]^{1/2}\right] \quad (11)$$
$$= \frac{g}{l}\left[(1+\frac{m_2}{m_1}) \pm \left[\frac{m_2}{m_1}(1+\frac{m_2}{m_1})\right]^{1/2}\right] \quad (12)$$
$$\lambda^{\pm} = \omega_0^2[1 + r \pm \sqrt{r(1+r)}] \quad (13)$$

where the substitutions are obvious.

Now let's find the eigenvectors $\vec{a}_{\pm}$ corresponding to the eigenvalues using $(\mathbf{V} - \lambda^{\pm}\mathbf{T})\vec{a}^{\pm} = \vec{0}$. We'll choose the first component of each $\vec{a}^{\pm}$ equal 1. then we solve the 2nd component of $a^{\pm}$ to get

$$-[1+r\pm\sqrt{r(1+r)}] + a_2^{\pm}(1 - [1+r\pm\sqrt{r(1+r)}]) = 0$$

so that

$$a_2^{\pm} = \frac{1+r\pm\sqrt{r(1+r)}}{1-[1+r\pm\sqrt{r(1+r)}]} = \mp\sqrt{\frac{1}{r}+1}$$

and

$$a^+ = \begin{pmatrix} 1 \\ -\sqrt{\frac{1}{r}+1} \end{pmatrix} \qquad a^- = \begin{pmatrix} 1 \\ +\sqrt{\frac{1}{r}+1} \end{pmatrix}$$

The a^+ solution has the bottom mass moving in the opposite direction from the top mass and the a^- solution has both masses moving in the same direction. (And yes, I really should normalize both vectors by dividing by a scale factor of $\sqrt{2+1/r}$.)

If $m_2 \ll m_1$, then $r \ll 1$ and

$$\lambda^{\pm} = \omega_0^2[1 + r \pm \sqrt{r}]$$

and $\lambda^+ \approx \lambda^-$.

Now let's consider the initial conditions with m_1 displaced slightly. In that case, $\theta_1(0)$ is non-zero, and $\theta_2(2) = \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$. We write

$$\theta_1(t) = C_+ a_1^+ \exp(-i\omega^+ t) + C_- a_1^- \exp(-i\omega^- t) \quad (14)$$
$$\theta_2(t) = C_+ a_2^+ \exp(-i\omega^+ t) + C_- a_2^- \exp(-i\omega^- t) \quad (15)$$

where $\omega^\pm = \sqrt{\lambda^\pm}$ and we want to solve for $C_\pm$. We also have the $\dot{\theta}_i$ constraints:

$$0 = \dot{\theta}_1(0) = C_+(-i\omega^+) + C_-(-i\omega^-) \quad (16)$$
$$0 = \dot{\theta}_2(0) = -C_+(-i\omega^+) + C_-(-i\omega^-) \quad (17)$$

where we used the fact that $a_1^+ = a_1^- = 1$ and $a_2^+ = -a_2^-$. These last two equations tell us that $C_\pm$ are real. This gives

$$C_+ + C_- = \theta_1(0) \quad (18)$$
$$C_+ a_2^+ + C_- a_2^- = 0 \quad (19)$$
$$\rightarrow C_+ = C_- = \frac{1}{2}\theta_1(0) \quad (20)$$

Thus our full solution for this initial condition is

$$\theta_1(t) = \frac{1}{2}\theta_1(0)(\cos\omega^+ t + \cos\omega^- t) \quad (21)$$
$$\theta_2(t) = \frac{1}{2}\theta_1(0)\sqrt{\frac{m_1}{m_2} + 1}(-\cos\omega^+ t + \cos\omega^- t) \quad (22)$$

If $m_1 \approx m_2$, then ω^+ is very close to ω^-. When t is such that $\cos\omega^+ t \approx \cos\omega^- t$ then m_1 moves a lot and m_2 moves a little (because of the cancelation in the $\theta_2(t)$ term). When t is such that $t(\omega^+ - \omega^-) \approx \pi$, then $\cos\omega^+ t \approx -\cos\omega^- t$ and m_2 moves a lot and m_1 moves little.

Question 2

> Three equal masses at the end of three equal length massless rods, each connected at one end at the origin and free to rotate. Find the normal modes.

Answer. We can choose to use either the angles of the rods, θ_i, or the differences of angles, $\phi ij = \theta_i - \theta_j$ for our coordinate system. If we use the differences, then we only two of them (since $\phi_{12} + \phi_{23} + \phi_{31} = 2\pi$ but we also need the angle of one of the rods to provide an absolute reference (e.g., $\theta_1, \phi_{12}, \phi_{23}$).

I will use θ_i because it is more intuitive to me. In this coordinate system,

$$T = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2)$$

and

$$V = \frac{1}{2}\kappa((\theta_1 - \theta_2 - \theta_0)^2 + (\theta_2 - \theta_3 - \theta_0)^2 + (\theta_3 - \theta_1 - \theta_0)^2) \quad (23)$$
$$= \frac{1}{2}\kappa(2\theta_1^2 + 2\theta_2^2 + 2\theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3 - 2\theta_3\theta_1 + 3\theta_0^2) \quad (24)$$

and we can ignore the θ_0 term in the potential because it is constant. We then get

$$\mathcal{L} = T - V = \frac{1}{2}\dot{\vec{\theta}}^T \mathbf{T} \dot{\vec{\theta}} - \frac{1}{2}\vec{\theta}^T \mathbf{V} \vec{\theta} \quad (25)$$

$$= \frac{1}{2} m l^2 \dot{\vec{\theta}}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{\vec{\theta}} - \frac{1}{2}\kappa \vec{\theta}^T \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \vec{\theta} \quad (26)$$

which leads to the eigenvector equation

$$0 = |\mathbf{V} - \lambda_i \mathbf{T}| \quad (27)$$

$$= \begin{vmatrix} 2\kappa - \lambda m l^2 & -\kappa & -\kappa \\ -\kappa & 2\kappa - \lambda m l^2 & -\kappa \\ -\kappa & -\kappa & 2\kappa - \lambda m l^2 \end{vmatrix} \quad (28)$$

$$= a^3 - a\kappa^2 + \kappa[(-\kappa)a - \kappa^2] - \kappa(\kappa^2 + \kappa a) \quad (29)$$

$$= a^3 - 3a\kappa^2 - 2\kappa^3 \quad (30)$$

where $a = 2\kappa - \lambda m l^2$. By inspection, $a = 2\kappa$ is a solution. Dividing $a - 2\kappa$ into the cubic gives its factors as

$$0 = (a - 2\kappa)(a + \kappa)^2.$$

The first eigenvalue is $a = 2\kappa$ gives

$$2\kappa - \lambda m l^2 = 2\kappa \to \lambda = 0$$

which gives eigenvector (by inspection)

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

which corresponds to a frequency $\omega = \sqrt{\lambda} = 0$ and all three rods rotating together. This is a rotation of the entire system.

The next two eigenvalues are degenerate and are

$$2k - \lambda m l^2 + k = 0$$

$$\lambda = \frac{3k}{ml^2}$$

and gives the eigenvector equation

$$(\mathbf{V} - \lambda \mathbf{T})\vec{a} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

If we choose $a_1 = 1$, then we get two solutions, $a_2 = 0$ and $a_3 = -1$ or $a_2 = -1$ and $a_3 = 0$. These correspond to breathing modes where two of the masses move closer to each other and then farther from each while the 3rd mass stays constant. The most general solution will be s supeposition of the overall rotation plus oscillations of 1 vs 2 and 1 vs 3.

To be complete, we should choose orthonormal modes such as $(1, -1, 0)/\sqrt{2}$ and $(1, 1, -2)/\sqrt{6}$. The first is the same as above, with one oscillating out of phase with 2. The second has 1 and 2 moving in phase with 3 moving out of phase with twice the amplitude.