OLD DOMINION UNIVERSITY

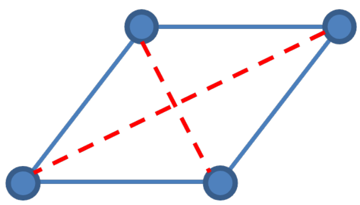
Midterm Solutions

Course: *Classical Mechanics (Physics 603), Prof. Weinstein*
Spring 2021

Question 1

Four massless rods of length L are hinged together at their ends to form a rhombus. A particle of mass m is attached at each joint. The opposite corners of the rhombus are joined by springs, each with spring constant k. In the equilibrium square configuration, the springs are unstretched. The motion is confined to a plane. Ignore the motion of the center of mass and assume that the system does not rotate.

- The system has a single degree of freedom. Starting from the eight degrees of freedom of four unconstrained masses, explain why this system only has one degree of freedom.
- Choose a suitable generalized coordinate and obtain the Lagrangian.
- Deduce the equation of motion
- Obtain the frequency of small oscillations about the equilibrium configuration.



Answer. 1) We number the masses starting with 1 in the lower left corner and proceeding clockwise. The mass 1 is unconstrained (2 dof). Mass 2 is a distance L from mass 1 and has 1 dof. Mass 4 is a distance L from mass 1 and therefore has 1 dof. Mass 3 is a distance L from both mass 2 and 4 and thus has 0 dof. This gives us four dof. Two concern the location of the center of mass. One concerns the rotation of the system. This leaves us with 1 dof.

2) Choose θ to be the lower left corner opening angle formed by mass 2 to mass 1 to mass 4. Then the positions of the four masses are

$$(x_1, y_1) = (0, 0) \tag{1}$$
$$(x_2, y_2) = (L\cos\theta, L\sin\theta) \tag{2}$$
$$(x_3, y_3) = (L + L\cos\theta, L\sin\theta) \tag{3}$$
$$(x_4, y_4) = (L, 0) \tag{4}$$

and length of the springs between masses 2 and 4 and between masses 1 and 3 are

$$l_{24} = \sqrt{(L(1-\cos\theta))^2 + L^2\sin^2\theta} \tag{5}$$

$$\begin{aligned}
&= \sqrt{2L^2 - 2L^2\cos\theta} && (6)\\
&= L\sqrt{2(1-\cos\theta)} && (7)\\
&= 2L\sin\theta/2 && (8)\\
l_{13} &= \sqrt{L^2(1+\cos\theta)^2 + L^2\sin^2\theta} && (9)\\
&= \sqrt{2L^2(1+\cos\theta} && (10)\\
&= 2L\cos\theta/2 && (11)
\end{aligned}$$

Now choose the CM as the origin. This gives new coordinates

$$\begin{aligned}
(x_1, y_1) &= (-\frac{1}{2}L(1+\cos\theta), -\frac{1}{2}L\sin\theta) && (12)\\
(x_2, y_2) &= (-\frac{1}{2}L(1-\cos\theta), \frac{1}{2}L\sin\theta) && (13)\\
(x_3, y_3) &= (\frac{1}{2}L(1+\cos\theta), \frac{1}{2}L\sin\theta) && (14)\\
(x_3, y_3) &= (\frac{1}{2}L(1-\cos\theta), -\frac{1}{2}L\sin\theta) && (15)
\end{aligned}$$

and velocities

$$\dot{x}_i = \pm\frac{1}{2}L\dot{\theta}\sin\theta \qquad \dot{y}_i = \pm\frac{1}{2}L\dot{\theta}\cos\theta$$

Thus

$$T = \frac{1}{2}m(\frac{1}{4}L^2\dot{\theta}^2)2(\sin^2\theta + \cos^2\theta) = \frac{1}{4}mL^2\dot{\theta}^2$$

where the factor of four comes from the four masses and

$$\begin{aligned}
V &= \frac{1}{2}k(2L\sin\theta/2 - \sqrt{2}L)^2 + \frac{1}{2}k(2L\cos\theta/2 - \sqrt{2}L)^2 && (16)\\
&= \frac{1}{2}kL^2(4\sin^2\theta/2 - 4\sqrt{2}\sin\theta/2 + 2 + 4\cos^2\theta/2 - 4\sqrt{2}\cos\theta/2 + 2) && (17)\\
&= \frac{1}{2}kL^2(8 - 4\sqrt{2}(\sin\theta/2 + \cos\theta/2)) && (18)
\end{aligned}$$

where $\sqrt{2}L$ is the unstretched length of the springs. Then the Lagrangian is

$$\mathcal{L} = \frac{1}{4}mL^2\dot{\theta}^2 + \frac{1}{2}kL^2 4\sqrt{2}(\sin\theta/2 + \cos\theta/2)$$

where I omitted the constant term in V.

3) The equations of motion are thus

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{1}{2}mL^2\ddot{\theta} = \frac{\partial\mathcal{L}}{\partial\theta} = \sqrt{2}kL^2(\cos\theta/2 - \sin\theta/2)$$

4) For small oscillations, let $\theta = \pi/2 + \theta'$. We need to expand $\cos\theta/2$ and $\sin\theta/2$ for small θ' using

$$f(x) = f(x_0) + f'(x_0)\delta x + \frac{1}{2}f''(x_0)(\delta x)^2$$

so that

$$\cos(\theta/2) = \cos(\pi/4 + \theta'/2) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\theta' - \frac{1}{4}\frac{1}{\sqrt{2}}\frac{1}{2}\theta'^2 + \dots \quad (19)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\theta' - \frac{1}{8\sqrt{2}}\theta'^2 \quad (20)$$

$$\sin(\theta/2) = \sin(\pi/4 + \theta'/2) = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\theta' - \frac{1}{8\sqrt{2}}\theta'^2 \quad (21)$$

and

$$\cos(\pi/4 + \theta'/2) - \sin(\pi/4 + \theta'/2) = -\frac{\theta'}{\sqrt{2}}.$$

Substituting in to the EOM we get

$$\frac{1}{2}mL^2\ddot{\theta}' = \sqrt{2}kL^2\left(-\frac{\theta'}{\sqrt{2}}\right) \quad (22)$$

$$\ddot{\theta}' = -\frac{2k}{m}\theta' \quad (23)$$

which by inspection gives a frequency of oscillation of

$$\omega = \sqrt{2k/m}$$

Question 2

A smooth wire is bent into the shape of a spiral helix. In cylindrical polar coordinates (ρ, ϕ, z) it is specified by equations $\rho = R\phi^2$ and $z = \lambda\phi^2$, where R and λ are constants and the z-axis is vertically up (and gravity is vertically down).

1. Using z as your generalized coordinate, write down the Lagrangian for a bead of mass m threaded on the wire.
2. Find the Lagrange equations of motion and find from it the expression for the bead's vertical acceleration $\ddot{z}$ as a function of z and $\dot{z}$.
3. Find the acceleration $\ddot{z}$ in two limits: (i) when $R \to 0$ but λ is fixed, and (ii) when $\lambda \to \infty$ but R is fixed. Discuss if your results for $\ddot{z}$ in these limits make sense.

Answer. 1) We start by writing the Lagrangian using all three corrdinates.

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - mgz$$

and then use $\rho = \frac{R}{\lambda}z$ to get

$$\mathcal{L} = \frac{1}{2}m(\frac{R^2}{\lambda^2}\dot{z}^2 + \frac{R^2}{\lambda^2}z^2\dot{\phi}^2 + \dot{z}^2) - mgz$$

and then use

$$\dot{z} = 2\lambda\phi\dot{\phi} \quad (24)$$

$$\dot{z}^2 = 4\lambda^2\phi^2\dot{\phi}^2 = 4\lambda z\dot{\phi}^2 \quad (25)$$

$$\dot{\phi}^2 = \frac{\dot{z}^2}{4\lambda z} \quad (26)$$

to get $\mathcal{L}$ solely in terms of z:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}m(\frac{R^2}{\lambda^2}\dot{z}^2 + \frac{R^2}{\lambda^2}z\frac{\dot{z}^2}{4\lambda z} + \dot{z}^2) - mgz] \qquad (27) \\
&= \frac{1}{2}m\dot{z}^2(1 + \frac{R^2}{\lambda^2} + \frac{R^2}{4\lambda^3}z) - mgz \qquad (28)
\end{aligned}$$

2) we can now find the EOM

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{z}}\right) = m\ddot{z}(1 + \frac{R^2}{\lambda^2} + \frac{R^2}{4\lambda^3}z) + m\dot{z}^2\frac{R^2}{4\lambda^3} = \frac{\partial\mathcal{L}}{\partial z} = \frac{1}{2}m\dot{z}^2\frac{R^2}{4\lambda^3} - mg$$

We can now solve this for $\ddot{z}$ to get this ugly thing:

$$\ddot{z} = \frac{-g - \dot{z}^2\frac{R^2}{8\lambda^3}}{1 + \frac{R^2}{\lambda^2} + \frac{R^2}{4\lambda^3}z}$$

which should be really easy to solve. Or not.

3) (i) as $R \to 0$, $\ddot{z} \to -g$. This makes sense because as $R \to 0$ the sprial helix becomes a straight vertical wire and the particle falls freely.

(ii) as $\lambda \to \infty$ for fixed R, we get exactly the same limit. This also makes sense, because the vertical motion dominates and ρ changes very slowly.

Question 3

Two bodies move under the influence of the central-force potential $V(r) = kr^\alpha$ where $\vec{r}$ is the relative coordinate and k and α are constants (ignore the center-of-mass motion).

1. Assume that $\vec{r}(t)$ is a solution to the equations of motion. Show that $\vec{r}'(t) = \lambda\vec{r}(\lambda^\sigma t)$ is also a solution to the equations of motion for any constant λ, provided the exponent σ is suitably chosen. What is the value of σ?
2. Apply the result from 1. to the cases $\alpha = 2$ (harmonic oscillator) and $\alpha = -1$ (Kepler problem). Comment on the results and on the properties you can derive for them.

(Qualifier Problem)
Hint: This does not require a lot of complicated math, just some clever argument. If you get stuck, email me – do NOT collaborate with your fellow students!

Answer. 1) We start by writing down the Lagrangian and the EOM:

$$\mathcal{L} = \frac{\mu}{2}\dot{r}^2 + \frac{\mu}{2}r^2\dot{\phi}^2 - kr^\alpha$$

and

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \mu\ddot{r} = \frac{\partial\mathcal{L}}{\partial r} = -k\alpha r^{\alpha-1} + \mu r\dot{\phi}^2$$

The Lagrangian is cyclic in ϕ so that

$$\mu r^2\ddot{\phi} = 0$$

We will assume that $\vec{r}(t)$ is a solution to the EOM:

$$\mu\ddot{r} = k\alpha r^{\alpha-1} + \mu r\dot{\phi}^2 \tag{29}$$

and see whether $\vec{r}'(t) = \lambda\vec{r}(\lambda^\sigma t)$ can also be a solution.

We will first calculate the time derivatives of r':

$$\frac{d\vec{r}'(t)}{dt} = \frac{d}{dt}(\lambda\vec{r}(\lambda^\sigma t)) = \lambda\lambda^\sigma\dot{\vec{r}}(\lambda^\sigma t) = \lambda^{\sigma+1}\dot{\vec{r}}(\lambda^\sigma t) \tag{30}$$

$$\frac{d^2\vec{r}'(t)}{dt^2} = \frac{d}{dt}\frac{d\vec{r}'(t)}{dt} = \lambda^{2\sigma+1}\ddot{\vec{r}}(\lambda^\sigma t) \tag{31}$$

Similarly

$$\dot{\phi}' = \lambda^\sigma\dot{\phi}$$

where the prefactor λ of $\vec{r}'$ does not affect $\dot{\phi}$ but the different time dependence does.

Now we can substitute these into Eq 29 and see whether $\vec{r}'$ can satisfy the EOM:

$$\mu\ddot{r}'(t) = k\alpha(r'(t))^{\alpha-1} + \mu r'(t)(\dot{\phi}'(t))^2 \tag{32}$$

$$\mu\lambda^{2\sigma+1}\ddot{r}(\lambda^\sigma t) = k\alpha\lambda^{\alpha-1}(r(\lambda^\sigma t))^{\alpha-1} + \mu\lambda r(\lambda^\sigma t)\lambda^{2\sigma}(\dot{\phi}(\lambda^\sigma t))^2 \tag{33}$$

$$\mu\ddot{r} = \lambda^{\alpha-1-2\sigma-1}r^{\alpha-1}k\alpha + \mu r\dot{\phi}^2 \tag{34}$$

where I dropped the time dependence in Eq 34. This reduces to Eq 29 if

$$\alpha - 1 - 2\sigma - 1 = 0$$

or if

$$\sigma = \frac{1}{2}\alpha - 1$$

2) Special cases: (i) Harmonic oscillator $\alpha = 2$: In this case $\sigma = 0$ and the only change in r' is multiplying the amplitude by λ. This simply shows that the time-dependence of the motion is independent of the amplitude, meaning for any solution $r(t)$, any multiple of that solution is also an allowed motion.

(ii) Gravity $\alpha = -1$: in this case $\sigma = -3/2$. This means that if we increase the orbital distance by a factor of λ, then we also increase the period by a factor of $\lambda^{3/2}$. In other words, $a \propto T^{3/2}$ or $a^2 \propto T^3$, which is Kepler's 3rd law.