Question 1

Four massless rods of length $L$ are hinged together at their ends to form a rhombus. A particle of mass $m$ is attached at each joint. The opposite corners of the rhombus are joined by springs, each with spring constant $k$. In the equilibrium square configuration, the springs are unstretched. The motion is confined to a plane. Ignore the motion of the center of mass and assume that the system does not rotate.

- The system has a single degree of freedom. Starting from the eight degrees of freedom of four unconstrained masses, explain why this system only has one degree of freedom.
- Choose a suitable generalized coordinate and obtain the Lagrangian.
- Deduce the equation of motion
- Obtain the frequency of small oscillations about the equilibrium configuration.

Answer. 1) We number the masses starting with 1 in the lower left corner and proceeding clockwise. The mass 1 is unconstrained (2 dof). Mass 2 is a distance $L$ from mass 1 and has 1 dof. Mass 4 is a distance $L$ from mass 1 and therefore has 1 dof. Mass 3 is a distance $L$ from both mass 2 and 4 and thus has 0 dof. This gives us four dof. Two concern the location of the center of mass. One concerns the rotation of the system. This leaves us with 1 dof.

2) Choose $\theta$ to be the lower left corner opening angle formed by mass 2 to mass 1 to mass 4. Then the positions of the four masses are

\[
\begin{align*}
(x_1, y_1) &= (0, 0) \\
(x_2, y_2) &= (L \cos \theta, L \sin \theta) \\
(x_3, y_3) &= (L + L \cos \theta, L \sin \theta) \\
(x_4, y_4) &= (L, 0)
\end{align*}
\]

and length of the springs between masses 2 and 4 and between masses 1 and 3 are

\[
l_{24} = \sqrt{(L(1 - \cos \theta))^2 + L^2 \sin^2 \theta}
\]
\[ L = \sqrt{2L^2 - 2L^2 \cos \theta} \]  
(6)
\[ = L \sqrt{2(1 - \cos \theta)} \]  
(7)
\[ = 2L \sin \theta / 2 \]  
(8)
\[ l_{13} = \sqrt{L^2(1 + \cos \theta)^2 + L^2 \sin^2 \theta} \]  
(9)
\[ = \sqrt{2L^2(1 + \cos \theta)} \]  
(10)
\[ = 2L \cos \theta / 2 \]  
(11)

Now choose the CM as the origin. This gives new coordinates
\[ (x_1, y_1) = (-\frac{1}{2}L(1 + \cos \theta), -\frac{1}{2}L \sin \theta) \]  
(12)
\[ (x_2, y_2) = (-\frac{1}{2}L(1 - \cos \theta), \frac{1}{2}L \sin \theta) \]  
(13)
\[ (x_3, y_3) = (\frac{1}{2}L(1 + \cos \theta), \frac{1}{2}L \sin \theta) \]  
(14)
\[ (x_3, y_3) = (\frac{1}{2}L(1 - \cos \theta), -\frac{1}{2}L \sin \theta) \]  
(15)

and velocities
\[ \dot{x}_i = \pm \frac{1}{2}L \dot{\theta} \sin \theta \quad \dot{y}_i = \pm \frac{1}{2}L \dot{\theta} \cos \theta \]

Thus
\[ T = \frac{1}{2}m(\frac{1}{4}L^2 \dot{\theta}^2)2(\sin^2 \theta + \cos^2 \theta) = \frac{1}{4}mL^2 \dot{\theta}^2 \]

where the factor of four comes from the four masses and
\[ V = \frac{1}{2}k(2L \sin \theta / 2 - \sqrt{2L})^2 + \frac{1}{2}k(2L \cos \theta / 2 - \sqrt{2L})^2 \]  
(16)
\[ = \frac{1}{2}kL^2(4 \sin^2 \theta / 2 - 4 \sqrt{2} \sin \theta / 2 + 2 + 4 \cos^2 \theta / 2 - 4 \sqrt{2} \cos \theta / 2 + 2) \]  
(17)
\[ = \frac{1}{2}kL^2(8 - 4 \sqrt{2}(\sin \theta / 2 + \cos \theta / 2)) \]  
(18)

where \( \sqrt{2L} \) is the unstretched length of the springs. Then the Lagrangian is
\[ \mathcal{L} = \frac{1}{4}mL^2 \dot{\theta}^2 + \frac{1}{2}kL^2 4\sqrt{2}(\sin \theta / 2 + \cos \theta / 2) \]

where I omitted the constant term in \( V \).

3) The equations of motion are thus
\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = \frac{1}{2}mL^2 \ddot{\theta} = \frac{\partial \mathcal{L}}{\partial \theta} = \sqrt{2}kL^2(\cos \theta / 2 - \sin \theta / 2) \]

4) For small oscillations, let \( \theta = \pi / 2 + \theta' \). We need to expand \( \cos \theta / 2 \) and \( \sin \theta / 2 \) for small \( \theta' \) using
\[ f(x) = f(x_0) + f'(x_0) \delta x + \frac{1}{2} f''(x_0) (\delta x)^2 \]
so that
\[
\cos(\theta/2) = \cos(\pi/4 + \theta'/2) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \frac{\theta'}{2} - \frac{1}{4} \frac{1}{\sqrt{2}} \frac{\theta'^2}{2} + \ldots \tag{19}
\]
\[
= \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} \theta' - \frac{1}{8\sqrt{2}} \theta'^2 \tag{20}
\]
\[
\sin(\theta/2) = \sin(\pi/4 + \theta'/2) = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \theta' - \frac{1}{8\sqrt{2}} \theta'^2 \tag{21}
\]
and
\[
\cos(\pi/4 + \theta'/2) - \sin(\pi/4 + \theta'/2) = -\frac{\theta'}{\sqrt{2}}.
\]
Substituting in to the EOM we get
\[
\frac{1}{2} mL^2 \ddot{\theta}' = \sqrt{2} kL^2 \left( -\frac{\theta'}{\sqrt{2}} \right) \tag{22}
\]
\[
\ddot{\theta}' = -\frac{2k}{m} \theta'
\]
which by inspection gives a frequency of oscillation of
\[
\omega = \sqrt{k/m}
\]

**Question 2**

A smooth wire is bent into the shape of a spiral helix. In cylindrical polar coordinates $(\rho, \phi, z)$ it is specified by equations $\rho = R\phi^2$ and $z = \lambda\phi^2$, where $R$ and $\lambda$ are constants and the $z$-axis is vertically up (and gravity is vertically down).

1. Using $z$ as your generalized coordinate, write down the Lagrangian for a bead of mass $m$ threaded on the wire.

2. Find the Lagrange equations of motion and find from it the expression for the bead’s vertical acceleration $\ddot{z}$ as a function of $z$ and $\dot{z}$.

3. Find the acceleration $\ddot{z}$ in two limits: (i) when $R \to 0$ but $\lambda$ is fixed, and (ii) when $\lambda \to \infty$ but $R$ is fixed. Discuss if your results for $\ddot{z}$ in these limits make sense.

**Answer.** 1) We start by writing the Lagrangian using all three corrdinates.
\[
\mathcal{L} = T - V = \frac{1}{2} m (\rho^2 + \rho^2 \dot{\phi}^2 + z^2) - mgz
\]
and then use $\rho = \frac{R}{\lambda} z$ to get
\[
\mathcal{L} = \frac{1}{2} m \left( \frac{R^2}{\lambda^2} z^2 + \frac{R^2}{\lambda^2} z^2 \dot{\phi}^2 + z^2 \right) - mgz
\]
and then use
\[
\dot{z} = 2\lambda \phi \dot{\phi} \tag{24}
\]
\[
\dot{z}^2 = 4\lambda^2 \phi^2 \dot{\phi}^2 = 4\lambda z \dot{\phi}^2 \tag{25}
\]
\[
\dot{\phi}^2 = \frac{\dot{z}}{4\lambda z} \tag{26}
\]
to get $\mathcal{L}$ solely in terms of $z$:

$$
\mathcal{L} = \frac{1}{2} m \left( \frac{R^2}{\lambda^2} \dot{z}^2 + \frac{R^2}{4\lambda^3} z\dot{z} + \dot{z}^2 \right) - mgz
$$

(27)

$$
= \frac{1}{2} mz^2 \left( 1 + \frac{R^2}{\lambda^2} + \frac{R^2}{4\lambda^3} z \right) - mgz
$$

(28)

2) we can now find the EOM

$$
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{z}} \right) = m\ddot{z} \left( 1 + \frac{R^2}{\lambda^2} + \frac{R^2}{4\lambda^3} z \right) + mz^2 \frac{R^2}{4\lambda^3} = \frac{\partial \mathcal{L}}{\partial z} = \frac{1}{2} mz^2 \frac{R^2}{4\lambda^3} - mg
$$

We can now solve this for $\ddot{z}$ to get this ugly thing:

$$
\ddot{z} = \frac{-g - \frac{z^2 R^2}{8\lambda^3}}{1 + \frac{R^2}{\lambda^2} + \frac{R^2}{4\lambda^3} z}
$$

which should be really easy to solve. Or not.

3) (i) as $R \to 0$, $\ddot{z} \to -g$. This makes sense because as $R \to 0$ the spiral helix becomes a straight vertical wire and the particle falls freely.

(ii) as $\lambda \to \infty$ for fixed $R$, we get exactly the same limit. This also makes sense, because the vertical motion dominates and $\rho$ changes very slowly.

**Question 3**

Two bodies move under the influence of the central-force potential $V(r) = kr^\alpha$ where $\vec{r}$ is the relative coordinate and $k$ and $\alpha$ are constants (ignore the center-of-mass motion).

1. Assume that $\vec{r}(t)$ is a solution to the equations of motion. Show that $\vec{r}'(t) = \lambda \vec{r}(\lambda^\sigma t)$ is also a solution to the equations of motion for any constant $\lambda$, provided the exponent $\sigma$ is suitably chosen. What is the value of $\sigma$?

2. Apply the result from 1. to the cases $\alpha = 2$ (harmonic oscillator) and $\alpha = -1$ (Kepler problem). Comment on the results and on the properties you can derive for them.

(Qualifier Problem)

Hint: This does not require a lot of complicated math, just some clever argument. If you get stuck, email me – do NOT collaborate with your fellow students!

**Answer.** 1) We start by writing down the Lagrangian and the EOM:

$$
\mathcal{L} = \frac{\mu}{2} \dot{\vec{r}}^2 + \frac{\mu}{2} \dot{\phi}^2 - kr^\alpha
$$

and

$$
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu \ddot{r} = \frac{\partial \mathcal{L}}{\partial r} = -kar^{\alpha-1} + \mu r \dot{\phi}^2
$$

The Lagrangian is cyclic in $\phi$ so that

$$
\mu r^2 \dot{\phi} = 0
$$
We will assume that \( \vec{r}(t) \) is a solution to the EOM:

\[
\mu \ddot{r} = k \alpha \sigma^{\alpha - 1} + \mu r \dot{\phi}^2
\]  

(29)

and see whether \( \vec{r}'(t) = \lambda \vec{r}(\lambda^\sigma t) \) can also be a solution.

We will first calculate the time derivatives of \( r' \):

\[
\frac{d\vec{r}(t)}{dt} = \lambda \lambda^{\sigma} \dot{r}(\lambda^\sigma t) = \lambda^{\sigma + 1} \dot{r}(\lambda^\sigma t)
\]

(30)

\[
\frac{d^2\vec{r}(t)}{dt^2} = \lambda^{2\sigma + 1} \ddot{r}(\lambda^\sigma t)
\]

(31)

Similarly

\[
\dot{\phi}' = \lambda^\sigma \dot{\phi}
\]

where the prefactor \( \lambda \) of \( r' \) does not affect \( \dot{\phi} \) but the different time dependence does.

Now we can substitute these into Eq 29 and see whether \( \vec{r}' \) can satisfy the EOM:

\[
\mu \ddot{r}' = k \alpha (r'(t))^{\alpha - 1} + \mu r'(t)(\dot{\phi}'(t))^2
\]

(32)

\[
\mu \lambda^{2\sigma + 1} r(\lambda^\sigma t) = k \alpha \lambda^{\alpha - 1}(r(\lambda^\sigma t))^{\alpha - 1} + \mu \lambda r(\lambda^\sigma t) \lambda^{2\sigma} (\dot{\phi}(\lambda^\sigma t))^2
\]

(33)

\[
\mu \ddot{r} = \lambda^{\alpha - 1 - 2\sigma - 1} r^{\alpha - 1} k \alpha + \mu r \dot{\phi}^2
\]

(34)

where I dropped the time dependence in Eq 34. This reduces to Eq 29 if

\[
\alpha - 1 - 2\sigma - 1 = 0
\]

or if

\[
\sigma = \frac{1}{2} \alpha - 1
\]

2) Special cases: (i) Harmonic oscillator \( \alpha = 2 \): In this case \( \sigma = 0 \) and the only change in \( r' \) is multiplying the amplitude by \( \lambda \). This simply shows that the time-dependence of the motion is independent of the amplitude, meaning for any solution \( r(t) \), any multiple of that solution is also an allowed motion.

(ii) Gravity \( \alpha = -1 \): in this case \( \sigma = -3/2 \). This means that if we increase the orbital distance by a factor of \( \lambda \), then we also increase the period by a factor of \( \lambda^{3/2} \). In other words, \( a \propto T^{3/2} \) or \( a^2 \propto T^3 \), which is Kepler’s 3rd law.