1 Lecture Notes Week 1

1.1 Introduction

With the exception of Thermodynamics and Statistical Mechanics, which suffer from limited information, all physical theories may be classified according to whether or not they are special relativistic (Rel), whether or not they are quantized, and, finally, the number of degrees of freedom (DoF) that they have (finite or infinite). (see Table 1)

Table 1: Classification of Physical Theories

		Finite DoF	Infinite Dof
Rel	Q	Rel QM	Rel QFT
	Non Q	Special Rel	EM
Non Rel	Q	QM	QFT
	Non Q	Classical	Fluids

1.2 Mechanics of a Single Particle

1.2.1 Motion of a Single Particle

Consider a particle of constant mass m at a position $\boldsymbol{r}$ in an inertial (non-accelerating) frame of reference. The velocity of the particle is given by the time derivative of the position $\boldsymbol{v} = \dot{\boldsymbol{r}}$, and its momentum is simply $\boldsymbol{p} = m\boldsymbol{v}$. If $\boldsymbol{F}$ is the force exerted on the particle and $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}$ is its acceleration, then Newton's Second Law (in linear form) gives its equation of motion

$$\begin{aligned}\boldsymbol{F} &= \dot{\boldsymbol{p}} \\ &= \frac{d}{dt}(m\boldsymbol{v}) \\ &= m\dot{\boldsymbol{v}} \\ &= m\boldsymbol{a} \qquad (1)\end{aligned}$$

Therefore, if $\boldsymbol{F} = \boldsymbol{0}$, then the momentum $\boldsymbol{p}$ is conserved.

Similarly, the angular momentum of the particle *about a specific origin* is defined to be $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, and the torque *about the same origin* exerted on the particle is defined to be $\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}$. So, Newton's Second Law (in rotational form) follows and gives its equation of motion

$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{r} \times \boldsymbol{F} \\ &= \boldsymbol{r} \times \dot{\boldsymbol{p}} \\ &= \boldsymbol{r} \times \frac{d}{dt}(m\boldsymbol{v}) \\ &= \boldsymbol{v} \times (m\boldsymbol{v}) + \boldsymbol{r} \times \frac{d}{dt}(m\boldsymbol{v}) \\ &= \dot{\boldsymbol{r}} \times (m\boldsymbol{v}) + \boldsymbol{r} \times \frac{d}{dt}(m\boldsymbol{v}) \\ &= \frac{d}{dt}(\boldsymbol{r} \times (m\dot{\boldsymbol{v}})) \\ &= \dot{\boldsymbol{L}} \end{aligned} \tag{2}$$

Therefore, if $\boldsymbol{N} = \boldsymbol{0}$, then the angular momentum $\boldsymbol{L}$ is conserved.

1.2.2 Work and Energy

The work done on a particle by a force $\boldsymbol{F}$ as it moves from position $\boldsymbol{r}_1$ to position $\boldsymbol{r}_2$ along some given path is

$$\begin{aligned} W_{12} &= \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{s} \\ &= m \int_{t_1}^{t_2} \boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt} dt \\ &= \frac{1}{2} m \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \frac{d}{dt}\left(v^2\right) dt \\ &= \frac{1}{2} m \left(v_2^2 - v_1^2\right) \end{aligned} \tag{3}$$

Noting that the kinetic energy is $T = \frac{1}{2}mv^2$, we have the so-called work-energy theorem: $W_{12} = T_2 - T_1$.

When W_{12} is path independent, then $\boldsymbol{F}$ is a conservative force. In this case, along any closed path,

$$\begin{aligned} W_{12} &= \oint_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{s} \\ &= 0 \end{aligned}$$

This is logically equivalent to the existence of a potential $V(\boldsymbol{r})$, which is unique up to addition of a constant C and satisfies

$$\boldsymbol{F} = -\boldsymbol{\nabla} V(\boldsymbol{r}) \tag{4}$$

Note that energy is only conserved if the potential V does not explicitly depend on the time t. If this is the case, and the potential does *not* depend explicitly on time, then

$dW + dV = 0$ which implies upon integrating that $T + V = E$ with E constant.

On the other hand, in the case in which the potential does depend explicitly on time, $V = V(\boldsymbol{r}, t)$ so that

$$\begin{aligned} dW &= \boldsymbol{F} \cdot d\boldsymbol{s} \\ &= -\frac{\partial V}{\partial s} ds \\ &\neq -dV \\ &= -\frac{\partial V}{\partial s} ds - \frac{\partial V}{\partial t} dt \end{aligned}$$

1.3 Mechanics of a System of Particles

Consider a system of N particles and suppose the i^{th} particle has constant mass m_i and position $\boldsymbol{r_i}$ in an inertial (non-accelerating) frame of reference. Its velocity is given by the time derivative of its position $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$, and its momentum is simply $\boldsymbol{p}_i = m\boldsymbol{v}_i$. If $\boldsymbol{F}_{ij}$ is the force exerted on the i^{th} particle by the j^{th} and $\boldsymbol{F}_i^{(\text{ext})}$ is the net external force on the i^{th} particle $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}$ is its acceleration, then Newton's Second Law (in linear form) gives its equation of motion considering the total force on it

$$\sum_j \boldsymbol{F}_{ij} + \boldsymbol{F}_i^{(\text{ext})} = \dot{\boldsymbol{p}}_i \tag{5}$$

Assuming the weak version of Newton's Third Law (action/reaction), $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$, and defining the total external force on the system $\boldsymbol{F}^{(ext)} = \sum_i \boldsymbol{F}_i^{(\text{ext})}$, the total mass $M = \sum_i m_i$, and the center of mass $\boldsymbol{R} = \sum_i m_i \boldsymbol{r}_i / M$ so that $\boldsymbol{P} = M\dot{\boldsymbol{R}}$, Newton's Second Law for the system reads

$$\boldsymbol{F}^{(\text{ext})} = \dot{\boldsymbol{P}} \tag{6}$$

Therefore, if $\boldsymbol{F}^{(\text{ext})} = \boldsymbol{0}$, then the total momentum $\boldsymbol{P}$ of the system is conserved.

The total angular momentum of the system *about a specific origin* is defined to be $\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i$, but its treatment is slightly more complicated. First, define $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$. Then, assuming a stronger version of Newton's Third Law (action/reaction), that the internal forces $\boldsymbol{F}_{ij} || \boldsymbol{r}_{ij}$, and taking the time derivative of $\boldsymbol{L}$,

$$\begin{aligned} \dot{\boldsymbol{L}} &= \sum_{i,j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} + \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})} \\ &= \frac{1}{2} \sum_{i,j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} + \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})} \\ &= \sum_{i,j} \boldsymbol{r}_{ij} \times \boldsymbol{F}_{ij} + \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})} \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})} \end{aligned}$$

So, further defining $\boldsymbol{N}_i^{(\text{ext})} = \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})}$ and $\boldsymbol{N}^{(\text{ext})} = \sum_i \boldsymbol{N}_i^{(\text{ext})}$, the rotational form of Newton's Second Law for a system of particles is

$$\boldsymbol{N}^{(\text{ext})} = \dot{\boldsymbol{L}} \tag{7}$$

Therefore, if $\boldsymbol{N}^{(\text{ext})} = \boldsymbol{0}$, then the total angular momentum $\boldsymbol{L}$ of the system is conserved.

Furthermore, defining the relative position of the i^{th} particle in the center of mass frame to be $\boldsymbol{r}_i' = \boldsymbol{r}_i - \boldsymbol{R}$ so that the relative velocity is $\boldsymbol{v}_i' = \boldsymbol{v}_i - \boldsymbol{V}$, the total angular momentum may be written as

$$\begin{aligned}
\boldsymbol{L} &= \sum_i (\boldsymbol{R} + \boldsymbol{r}_i') \times m_i (\boldsymbol{V} + \boldsymbol{v}_i') \\
&= \sum_i \boldsymbol{R} \times m_i \boldsymbol{V} + \sum_i \boldsymbol{r}_i' \times m_i \boldsymbol{v}_i' \\
&+ \sum_i \boldsymbol{R} \times m_i \boldsymbol{v}_i' + \sum_i \boldsymbol{r}_i' \times m_i \boldsymbol{V} \\
&= \sum_i \boldsymbol{R} \times m_i \boldsymbol{V} + \sum_i \boldsymbol{r}_i' \times m_i \boldsymbol{v}_i' \\
&= \boldsymbol{R} \times M \boldsymbol{V} + \sum_i \boldsymbol{r}_i' \times \boldsymbol{p}_i'
\end{aligned}$$

1.3.1 Work and Energy

The work done on the system by all net forces $\boldsymbol{F}_i$ on each in a system to move it from some initial configuration 1 to a final configuration 2 is

$$\begin{aligned}
W_{12} &= \int_1^2 \sum_i \boldsymbol{F}_i \cdot d\boldsymbol{s}_i \\
&= \sum_i m_i \int_{t_1}^{t_2} \boldsymbol{v_i} \cdot \frac{d\boldsymbol{v}_i}{dt} dt \\
&= \sum_i T_2^i - T_1^i
\end{aligned} \tag{8}$$

When $\boldsymbol{F}_i^{(\text{ext})}$ is a conservative force, it can be written as the gradient of some potential

$$\boldsymbol{F}_i^{(\text{ext})} = -\boldsymbol{\nabla} V_i^{(\text{ext})} \tag{9}$$

Similarly, when $\boldsymbol{F}_{ij}$ is a conservative force it can also be written as the gradient of some potential

$$\boldsymbol{F}_{ij} = -\boldsymbol{\nabla}_i V_{ij} \tag{10}$$

If the strong version of Newton's Third Law (action/reaction) is also assumed, then this potential has the form $V_{ij} = V_{ij}\left(|\boldsymbol{r}_i - \boldsymbol{r}_j|\right)$

If all of the internal and external forces are conservative, then the total potential of the system is

$$V = \sum_i V_i^{(\text{ext})} + \frac{1}{2}\sum_{i,j} V_{ij}$$

Then the net force on the i^{th} particle is simply

$$\boldsymbol{F}_i = -\boldsymbol{\nabla} V$$

1.4 Forces of Constraint

Suppose that the set of positions of the N particles $\{\boldsymbol{r}_1, ..., \boldsymbol{r}_N\}$ are not independent. Then they are constrained, and there exist forces of constraint.

We primarily focus on holonomic constraints, that is constraints that can be expressed as $f(\boldsymbol{r}_1, ..., \boldsymbol{r}_N, t) = 0$.

Examples of holonomic constraints include: a particle on the plane $x = 5$ or a particle on the ellipse $x^2/A^2 + y^2/B^2 = C^2$.

An example of a non-holonomic constraint is a particle inside of the circle $x^2 + y^2 < a^2$.

Note that we may not know what the forces are, but we do know what they do to the particles in the system. For instance, we may know that a constraint force confines a particle to a plane or a bead to a wire, but we may not know what the force actually is.

Since there are N particles in the system inhabiting three-dimensional space, there are $3N$ original position coordinates. If we have k constraints, then there we have $n = 3N - k$ degrees of freedom. And, each of our positions (in fact, each coordinate of each position) may be written as a functions of our choice in $n = 3N - k$ generalized coordinates: $\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, ..., q_n, t)$. These coordinates may not form 3-vectors and may not be lengths.

1.5 d'Alembert's Principle and Lagrange's Equations

Let the net force on the i^{th} particle be given by $\boldsymbol{F}_i = \boldsymbol{F}_i^a + \boldsymbol{f}_i$, where $\boldsymbol{F}_i^a$ is the applied force and $\boldsymbol{f}_i$ is the force of constraint, and consider a virtual infinitesimal displacement $\delta\boldsymbol{r}_i$ from some static equilibrium $\boldsymbol{F}_i = \boldsymbol{0}$. We want to choose systems in which constraint forces do no work. Then, since $\boldsymbol{F}_i \cdot \delta\boldsymbol{r}_i = 0$,

$$\begin{aligned}
\boldsymbol{F}_i \cdot \delta\boldsymbol{r}_i &= \boldsymbol{F}_i^a \cdot \delta\boldsymbol{r}_i + \boldsymbol{f}_i \cdot \delta\boldsymbol{r}_i \\
&= \boldsymbol{F}_i^a \cdot \delta\boldsymbol{r}_i \\
&= 0
\end{aligned}$$

For a dynamic equilibrium, we still choose systems in which the constraint forces do no work, so

$$\sum_i (\boldsymbol{F}_i - \dot{\boldsymbol{p}}_i) \cdot \delta \boldsymbol{r}_i = \sum_i (\boldsymbol{F}_i^a - \dot{\boldsymbol{p}}_i) \cdot \delta \boldsymbol{r}_i = 0 \tag{11}$$

Now, we transform to our choice of generalized coordinates so that $\boldsymbol{r}_i = \boldsymbol{r}_i (q_1, ..., q_n, t)$. Using the chain rule, the velocity and displacement are

$$\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t}$$

$$\delta \boldsymbol{r}_i = \dot{\boldsymbol{r}}_i = \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j + \frac{\partial \boldsymbol{r}_i}{\partial t}$$

This allows us to define a generalized force

$$Q_j = \sum_i \boldsymbol{F}_i^a \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}$$

From which it follows that

$$\sum_i \boldsymbol{F}_i^a \cdot \delta \boldsymbol{r}_i = \sum_i \sum_j \boldsymbol{F}_i^a \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j \tag{12}$$

Thus, $Q_j q_j$ has units of work.

On the other hand, note the relations

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) = \sum_k \frac{\partial^2 \boldsymbol{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \boldsymbol{r}_i}{\partial q_j \partial t} = \frac{\partial \boldsymbol{v}_i}{\partial q_j}$$

$$\frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j} = \sum_k \frac{\partial \boldsymbol{r}_i}{\partial q_k} \delta_{jk} = \frac{\partial \boldsymbol{r}_i}{\partial q_j}$$

These imply that

$$\begin{aligned}
\sum_i m_i \ddot{r}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} &= \sum_i \left[\frac{d}{dt}\left(m_i \dot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) - m_i \dot{\boldsymbol{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right)\right] \\
&= \sum_i \left[\frac{d}{dt}\left(m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_j}\right) - m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial q_j}\right] \\
&= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right) - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right) \\
&= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}
\end{aligned} \tag{13}$$

From which it follows

$$\begin{aligned}\sum_i \dot{\boldsymbol{p}}_i \cdot \delta \boldsymbol{r}_i &= \sum_i \sum_j m_i \ddot{\boldsymbol{r}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \\ &= \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j\end{aligned}$$

But, then combining equations 12 and 13 and substituting into equation 11,

$$\sum_i (\boldsymbol{F}_i - \dot{\boldsymbol{p}}_i) \cdot \delta \boldsymbol{r}_i = \sum_j \left[Q_j - \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) \right] \delta q_j = 0$$

We arrive at Lagrange's equations. For, $j = 1, ..., n$,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \tag{14}$$

If we know that the applied forces on each particle are conservative, then for the total potential of the system V, the generalized force for $j = 1, ..., n$ may be written

$$Q_j = \sum_i \boldsymbol{F}_i^a \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\sum_i \boldsymbol{\nabla}_i V \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

And, if we further know that V does not depend on any velocities, then for $j = 1, ..., n$,

$$\frac{\partial V}{\partial \dot{q}_j} = 0$$

Then finally defining the Lagrangian of the system to be $\mathcal{L} = T - V$, we have the final form of Lagrange's equations, for $j = 1, ..., n$,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \tag{15}$$