Fundamentals of Engineering Review

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FE MATH REVIEW
ALGEBRA AND TRIG

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Introduction - Algebra

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- Cartesian Coordinates
- Lines and Linear Equations
- Quadratics
- Logs and exponents
- Inequalities and absolute values
- Simultaneous Equations
- Conic Sections
- Complex Numbers

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Cartesian Coordinates

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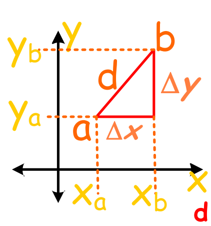

Pythagorean Theorem

$$d^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\Delta x = x_b - x_a$$

$$\Delta y = y_b - y_a$$

$$d = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

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Slopes of Lines

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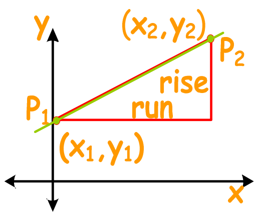

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$$

$$= \frac{y_2 - y_1}{x_2 - x_1} = m$$

move from $P_1(x_1, y_1)$ to $P(x, y)$

$$m = \frac{y - y_1}{x - x_1}$$

Equation of a Line

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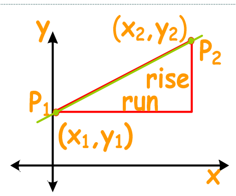

Using the equation for the slope of a line between 2 points, we can develop the general equation for a line.

$$m = \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \boxed{y = mx + b}$$

m = slope

b = y intercept

Parallel lines: $m_1 = m_2$

Perpendicular lines: $m_1 = -\dfrac{1}{m_2}$

Linear equation: $0 = Ax + By + C$

As long as the order of both x and y is 1 the equation is that of a straight line. If the order changes then it begins to take on different shapes. If y is of order 1 and x is of order 2 then the equation is **Quadratic** and will generate a **parabola**.

Example

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The equation $y = a^3 + a^2x$
where x and y are variables, and a is a constant.

Which of the following are represented by the equation?
(A) a 3rd order polynomial (B) a quadratic equation
(C) a straight line (D) acceleration

Answer:
The terms with a in them are constants so they don't determine the degree of the function. The implied Degree of both variables is a 1. The answer is (C).

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Example

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What is the slope of the line: $2y = 2x + 4$

Solution

Converting to the general form we get

$0 = 2y - 2x - 4$

$2y = 2x + 4$

$y = x + 2$

$\boxed{\text{slope} = 1}$

$\boxed{y \text{ intercept} = 2}$

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Example:

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What is the slope of the line perpendicular to $y = 16x - 4$

Answer

If line A is perpendicular to the line B, then $m_1 = -1/m_2$

$m_2 = 16$

$m_1 = -1/m_2 = \boxed{-1/16}$

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Example

What is the distance between the point $(-3,0)$ and the y intercept of the line $y + 4x - 4 = 0$?

Answer :

$$y = -4x + 4 \therefore b = (0,4)$$

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

$$= \sqrt{(4-0)^2 + (0 - [-3])^2}$$

$$= \sqrt{4^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = \boxed{5}$$

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Solving Quadratic Equations

- Using the Quadratic Formula
- Factoring
- Complete the squares
- Matrix methods (later)

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Quadratic Equations: Formula

$ax^2 + bx + c = 0$ (y has been set to 0)

Method 1: Quadratic Formula

$$x = \boxed{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Two imaginary solutions (no real solutions): $4ac > b^2$

One solution: $4ac = b^2$

Two real solutions: $4ac < b^2$

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For Example:

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$$0 = x^2 + 3x + 2$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{9 - 8}}{2}$$

$$= \frac{-3 \pm \sqrt{1}}{2} = \frac{-3 \pm 1}{2}$$

$$\Rightarrow \boxed{x = \frac{-4}{2}, \frac{-2}{2} \quad \text{or} \quad -2, -1}$$

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Example:

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Will the roots of the equation below be real, complex, or imaginary?

$$-1 = 50x^2 + 5(x - 2)^2$$

where x is a real valued variable.

$$0 = 50x^2 + 5(x^2 - 4x + 4) + 1$$

$$0 = 55x^2 - 20x + 21$$

$$\text{Roots} = \frac{-20 \pm \sqrt{20^2 - 4(55)(21)}}{2(55)} = \frac{-20 \pm \sqrt{400 - 4,620}}{110}$$

answer will be complex

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Quadratics: Factoring

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$$x^2 + 3x + 2 = (x + 2)(x + 1)$$

$$0 = x_1 + 2 \quad \therefore \quad \boxed{x_1 = -2}$$

$$0 = x_2 + 1 \quad \therefore \quad \boxed{x_2 = -1}$$

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Quadratics: Completing the square

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$0 = x^2 + 3x + 2$

1st, make sure that the coefficient of x^2 is 1.

2nd, take the coefficient of x and divide it by 2

$$-2 + \left(\frac{3}{2}\right)^2 = x^2 + 3x + \left(\frac{3}{2}\right)^2$$

3rd, add the resulting number to both sides of the equation.

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Completing the square (cont)

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$$-2 + \left(\frac{3}{2}\right)^2 = x^2 + 3x + \left(\frac{3}{2}\right)^2$$

$$\sqrt{\frac{1}{4}} = \sqrt{\left(x + \frac{3}{2}\right)^2}$$

$$\pm\frac{1}{2} = x + \frac{3}{2}$$

$$\boxed{x = -\tfrac{3}{2} \pm \tfrac{1}{2} = -1 \quad \text{or} \quad -2}$$

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Logarithms and Exponentials

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$$y = \log_a x \Leftrightarrow a^y = x$$

$$a = \text{base}$$

$$\log_e x = \ln x \qquad \underbrace{(e = 2.71828\ldots)}_{\text{irrational number}}$$

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Logarithms and Exponentials (a few properties)

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$$e^{a+b} = e^a \bullet e^b$$
$$e^{-a} = \frac{-1}{e^a}$$
$$\left(e^a\right)^b = e^{a \bullet b}$$

$$\ln\left(a \bullet b\right) = \ln\left(a\right) + \ln\left(b\right)$$
$$\ln\left(\frac{a}{b}\right) = \ln\left(a\right) - \ln\left(b\right)$$
$$\ln\left(a^b\right) = b\ln\left(a\right)$$
$$\log_a\left(b\right) = \frac{\ln\left(b\right)}{\ln\left(a\right)}$$

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Log/ln example

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$$\log_3\left(7\right) = \frac{\ln 7}{\ln 3} = \frac{1.946}{1.099} = \boxed{1.771}$$

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Absolute values

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$|x|$ = always positive

suppose $a < b$

$c \bullet a < c \bullet b$ if c is positive

$c \bullet a > c \bullet b$ if c is negative

Note that when you multiply both sides by a negative number the inequality changes.

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Example

solve $|x-2| < 5$

$|x-2| = x-2$ if $x-2 > 0$

so, $x-2 < 5 \Rightarrow \boxed{x = 7}$

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Example

solve $|x-2| < 5$

$|x-2| = x-2$ if $x-2 > 0$

so, $x-2 < 5 \Rightarrow \boxed{x = 7}$

$|x-2| = -(x-2)$ if $x-2 < 0$

so, $-(x-2) < 5$

$x-2 > -5$

$\Rightarrow \boxed{x = -3}$

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Example

solve $|x-2| < 5$

$|x-2| = x-2$ if $x-2 > 0$

so, $x-2 < 5 \Rightarrow \boxed{x = 7}$

$|x-2| = -(x-2)$ if $x-2 < 0$

so, $-(x-2) < 5$

$x-2 > -5$

$\Rightarrow \boxed{x = -3}$

solution : $x > -3$ and $x < 7 \Rightarrow \boxed{-3 < x < 7}$

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Common Log Example

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What is the common log of $(1000)^4$?

Identity : $\log_{10} 10^n = n$

$(1000)^4 = (10^3)^4 = 10^{12}$

$\log_{10} 10^{12} = 12 \log_{10} 10 = 12(1) = \boxed{12}$

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Natural Log Example

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What is the natual log of e^{x-y} ?

identity : $\ln e^c = c$

$\ln\left(e^{x-y}\right) = \boxed{x - y}$

Note that the natural log is sometimes called the Naperian logarithm

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Conic Sections

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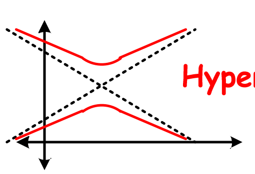
$Ax^2 + By^2 + Cx + Dy + E = 0$

A or B equals 0 (but not both)



Hyperbola

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Conic Sections (continued)

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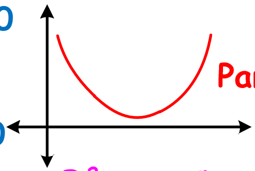
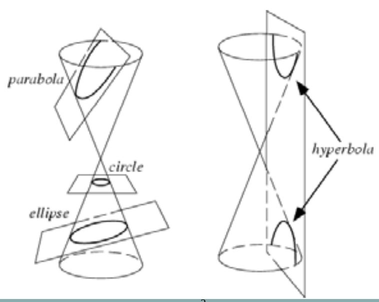
$$0 = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F$$

A or C = 0 (but not both) and B = 0



$$B^2 - AC = 0$$

Why are these called Conic Sections?



Parabola

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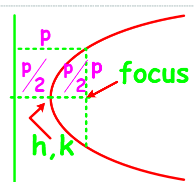
If opens left or right the equation will fit

$$(y - k)^2 = 2p(x - h)$$



$p > 0$ means opens right, $p < 0$ means opens left

center at (h,k), focus at $\left(h + \frac{p}{2}, k\right)$

direction at $\left(x = h - \frac{p}{2}\right)$

Parabola (cont)

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If opens up or down the equation will fit

$$2p\left(y-k\right)=\left(x-h\right)^2$$

$p > 0$ means opens up, $p < 0$ means opens down

center at $\left(h,k\right)$, focus at $\left(h,k+p/2\right)$

direction at $\left(y=k-p/2\right)$

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Ellipse

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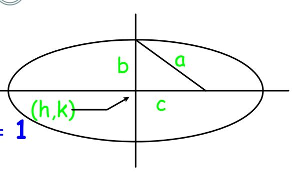

$B^2-AC<0$

$$\frac{\left(x-h\right)^2}{a^2}+\frac{\left(y-k\right)^2}{b^2}=1$$

center at $\left(h,k\right)$,

width $=2a\ \left(x\text{ - axis}\right)$

height $h=2b\ \left(y\text{ - axis}\right)$

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Example

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What shape does this equation represent?

$$\left(y-2\right)\left(y+2\right)=14x-12$$

$0=Ax^2+2Bxy+Cy^2+2Dx+2Ey+F$

$0=y^2-4-14x+12$

in parabola, either A or $C=0$, not both AND $B=0$

$0=0x^2+2\left(0\right)xy+Cy^2+2\left(7\right)x+2\left(0\right)y+F$

$0=1y^2+14x+8$

$\boxed{B^2-AC=0\ \therefore\ \text{parabola}}$

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Complex Numbers

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'j' and 'i' are used to represent complex numbers.
'i' is normally used in math and physics while
'j' is normally used in engineering (specifically electrical)

$$i = \sqrt{-1} \qquad i^2 = -1 \qquad i^3 = -i$$
$$j = \sqrt{-1} \qquad j^2 = -1 \qquad j^3 = -i$$

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Complex Numbers Example

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$$(4 + j7) + (6 + j9) = ?$$
$$(4 + 6) + (j7 + j9) = \boxed{10 + j16}$$

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Complex Numbers Example

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Complete the following math equation.
Your final answer should be in Polar Notation.

$$(2 - i2) \bullet \sqrt{32e^{i\frac{\pi}{4}}}$$

$$= \left[\sqrt{2^2 + (-2)^2} \measuredangle \tan^{-1}\left(\frac{-2}{2}\right)\right] \bullet \sqrt{32 \measuredangle \left(\frac{\pi}{4}\text{rad}\right)\left(\frac{180°}{\pi}\right)}$$

$$= 2.82 \measuredangle -45° \bullet \sqrt{32 \measuredangle 45°}$$

$$= \boxed{15.952 \measuredangle 0°}$$

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Complex Conjugates

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$$z^* = x - jy \quad \text{if} \quad z = x + jy$$

$$z \bullet z^* = x^2 + y^2$$

Complex Conjugate example

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solve : $\dfrac{2 + j}{3 + 2j}$

$$\frac{(2 + j)(3 - 2j)}{(3 + 2j)(3 - 2j)} = \frac{6 - 4j + 3j - 2j^2}{3^2 + 2^2}$$

$$= \frac{6 - j - 2(-1)}{9 + 4}$$

$$= \frac{6 - j + 2}{13} = \boxed{\frac{8 - j}{13}}$$

Complex Number Conversions

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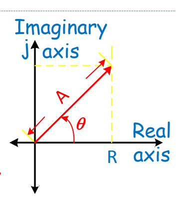

$$A\measuredangle\theta \quad = \quad R + jI$$

$$A = \sqrt{R^2 + I^2} \quad \Leftrightarrow \quad R = A\cos(\theta)$$

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \qquad I = A\sin(\theta)$$

Example

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$$R + jI = 3 + j4$$

$$A = \sqrt{3^2 + 4^2}$$

$$= \sqrt{9 + 16} = \sqrt{25} = \boxed{5}$$

$$\measuredangle\theta = \measuredangle \tan^{-1}\left(\frac{I}{R}\right)$$

$$= \measuredangle \tan^{-1}\left(\frac{4}{3}\right) = \boxed{53.13^\circ}$$

$$\overline{A} = A\measuredangle\theta = \boxed{5\measuredangle 53.13^\circ}$$

Simultaneous Equations

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A set of equations can be solved simultaneously if the number of unknowns is equal to the number of equations. There are several ways to solve them including via Matrix methods which will be discussed later.

Simultaneous Equations

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Method 1. Solve one equation for one variable and then substitute it into the next equation.

solve : $\boxed{\begin{array}{l} 7 = 2x + y \\ -1 = x - y \end{array}}$

$$-1 = x - y \Rightarrow \boxed{x = y - 1}$$

Simultaneous Equations

43

Method 1. Solve one equation for one variable and then substitute it into the next equation.

solve : $\boxed{\begin{array}{l} 7 = 2x + y \\ -1 = x - y \end{array}}$

$-1 = x - y \Rightarrow \boxed{x = y - 1}$

substituting :

$7 = 2(y - 1) + y$

$7 = 2y + y - 2$

$9 = 3y \Rightarrow \boxed{y = 3}$

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Simultaneous Equations

44

Method 1. Solve one equation for one variable and then substitute it into the next equation.

solve : $\boxed{\begin{array}{l} 7 = 2x + y \\ -1 = x - y \end{array}}$

$-1 = x - y \Rightarrow \boxed{x = y - 1}$

substituting :

$7 = 2(y - 1) + y$ and $-1 = x - y$

$7 = 2y + y - 2$ $\quad -1 = x - 3$

$9 = 3y \Rightarrow \boxed{y = 3}$ $\quad x = \boxed{2}$

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Simultaneous Equations

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2nd method: Multiply the equations by numbers such that when added together, only one variable will be left.

Solve :
$$\begin{array}{rcl} 7 &=& 2x + y \\ \underline{-1} &\underline{=}& \underline{x - y} \end{array}$$

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Simultaneous Equations

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2nd method: Multiply the equations by numbers such that when added together, only one variable will be left.

Solve :
$$\begin{aligned} 7 &= 2x + y \\ -1 &= x - y \end{aligned}$$

Multiply both sides of second eqn by -2
$$\begin{aligned} 7 &= 2x + y \\ (-2)\quad -1 &= x - y \end{aligned}$$

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Simultaneous Equations

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2nd method: Multiply the equations by numbers such that when added together, only one variable will be left.

Solve :
$$\begin{aligned} 7 &= 2x + y \\ -1 &= x - y \end{aligned}$$

Multiply both sides of second eqn by -2
$$\begin{aligned} 7 &= 2x + y \\ (-2)\quad -1 &= x - y \end{aligned}$$
$$\begin{aligned} 7 &= 2x + y \\ 2 &= -2x + 2y \end{aligned}$$

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Simultaneous Equations

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2nd method: Multiply the equations by numbers such that when added together, only one variable will be left.

Solve :
$$\begin{aligned} 7 &= 2x + y \\ -1 &= x - y \end{aligned}$$

Multiply both sides of second eqn by -2
$$\begin{aligned} 7 &= 2x + y \\ (-2)\quad -1 &= x - y \end{aligned}$$
$$\begin{aligned} 7 &= 2x + y \\ 2 &= -2x + 2y \\ 9 &= 0x + 3y \end{aligned}$$

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Simultaneous Equations

49

2nd method: Multiply the equations by numbers such that when added together, only one variable will be left.

$$\begin{array}{rcrcr} 7 & = & 2x & + & y \\ 2 & = & -2x & + & 2y \\ \hline 9 & = & 0x & + & 3y \end{array}$$

$3y = 9$ and $-1 = x - 3$

$y = \boxed{3}$ $\quad x = \boxed{2}$

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Previous Example Modified

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A second look at the equations with a clearer mind:

The equations already are set up such that if they were added y would fall out without any help.

$$\begin{array}{rcrcr} 7 & = & 2x & + & y \\ -1 & = & x & - & y \\ \hline \end{array}$$

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Previous Example Modified

51

A second look at the equations with a clearer mind:

The equations already are set up such that if they were added y would fall out without any help.

$$\begin{array}{rcrcr} 7 & = & 2x & + & y \\ -1 & = & x & - & y \\ \hline 6 & = & 3x & & \end{array}$$

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Previous Example Modified

52

A second look at the equations with a clearer mind:

The equations already are set up such that if they were added y would fall out without any help.

$$\begin{array}{rcl} 7 & = & 2x + y \\ -1 & = & x - y \\ \hline 6 & = & 3x \end{array}$$

$$x = \boxed{2}$$

$$-1 = x - y$$
$$-1 = 2 - y$$
$$y = \boxed{3}$$

Trigonometry basics

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- Unit Circle
- Triangles
- A few identities

The Unit Circle

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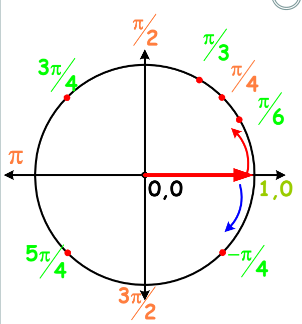
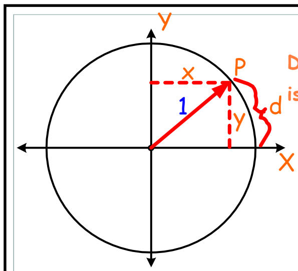


$r = 1$

$C = 2\pi r = 2\pi(1) = 2\pi$

Units = radians



Distance (d) from point (1,0) is point P on the circle.

$\sin(d) = y$ coordinate

$\cos(d) = x$ coordinate

$$\tan(d) = \frac{\sin(d)}{\cos(d)} = \frac{y}{x} \qquad \sec(d) = \frac{1}{\cos(d)} = \frac{1}{x}$$

$$\cot(d) = \frac{1}{\tan(d)} = \frac{x}{y} \qquad c\sec(d) = \frac{1}{\sin(d)} = \frac{1}{y}$$

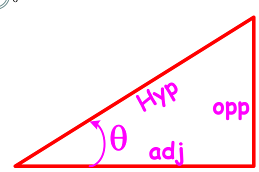
Triangle point of view

$$\sin\theta = \frac{opp}{hyp}$$

$$\cos\theta = \frac{adj}{hyp}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opp}{adj}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{adj}{opp}$$



$$\sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj}$$

$$\csc = \frac{1}{\sin\theta} = \frac{hyp}{opp}$$

A few identities

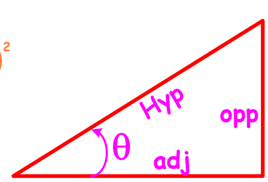
$$c^2 = x^2 + y^2$$

$$(hyp)^2 = (adj)^2 + (opp)^2$$

$$1 = \left(\frac{adj}{hyp}\right)^2 + \left(\frac{opp}{hyp}\right)^2$$

$$1 = \cos^2\theta + \sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$



$$\sin(\theta_1 \pm \theta_2) = \sin(\theta_1)\cos(\theta_2) \pm \cos(\theta_1)\sin(\theta_2)$$