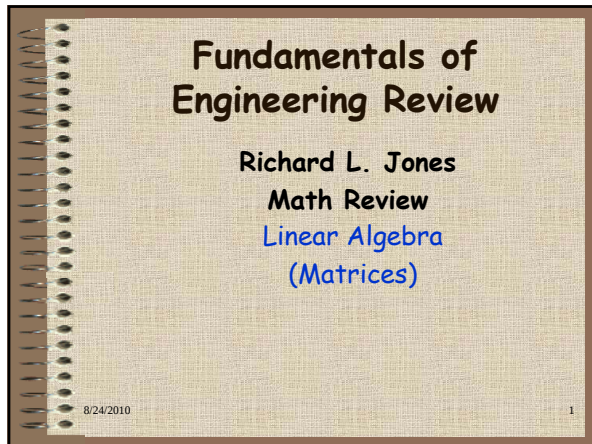
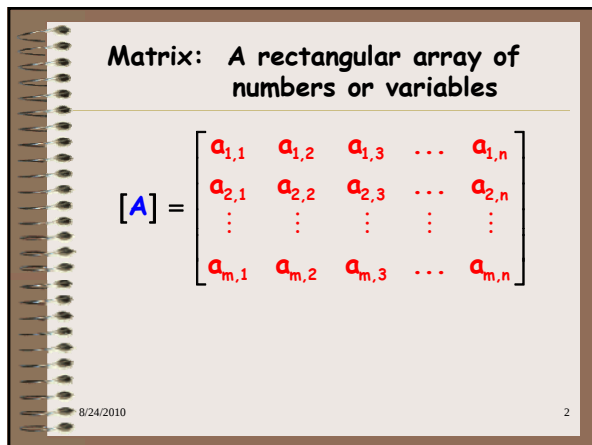
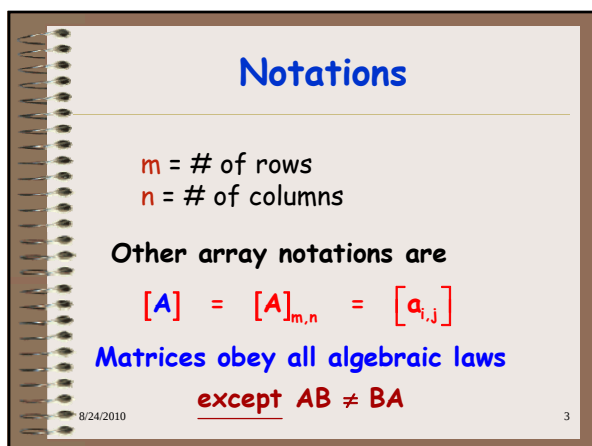








Example

$$[A] = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 4 & 6 \end{bmatrix} = [A]_{2,3}$$

$$[B] = \begin{bmatrix} 2 & 1 \\ 7 & -4 \\ 3 & 1 \end{bmatrix} = [B]_{3,2}$$

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Square Matrices

If $m = n$, the matrix is said to be "Square"

$$[C] = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 6 & 1 \\ -5 & 2 & 1 \end{bmatrix} = [C]_{3,3}$$

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Vector Dot Product

What is the angle between the vectors $\hat{F}_1$ and $\hat{F}_2$?

$$\hat{F}_1 = 5\hat{i} + 4\hat{j} + 6\hat{k} \quad \text{and} \quad \hat{F}_2 = 4\hat{i} + 10\hat{j} + 7\hat{k}$$

$$\begin{aligned} \cos \theta &= \frac{\hat{F}_1 \cdot \hat{F}_2}{|\hat{F}_1| |\hat{F}_2|} \\ &= \frac{5(4) + 4(10) + 6(7)}{\sqrt{5^2 + 4^2 + 6^2} \sqrt{4^2 + 10^2 + 7^2}} \\ &= \frac{20 + 40 + 42}{\sqrt{25 + 16 + 36} \sqrt{16 + 100 + 49}} \\ &= \frac{102}{\sqrt{77} \sqrt{165}} = \frac{102}{112.7165} = .905 \end{aligned}$$

$$\cos^{-1}(.905) = 25.18^\circ$$

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Column & Row Vectors

A matrix of ORDER $m \times 1$ is a column vector

Examples :

$$\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{Column vector of ORDER 3}$$

$$\bar{i} = [i_1 \ i_2 \ i_3 \ i_4] \quad \text{Row vector of ORDER 4}$$

A matrix of ORDER $1 \times n$ is a row vector

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Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+2 \\ 3+3 & 4+4 \end{bmatrix} \\ = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

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Matrix Dot Product Example

$$4 \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 \cdot 2 & 4 \cdot 3 \\ 4 \cdot 4 & 4 \cdot 5 \end{bmatrix} \\ = \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix}$$

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Matrix Multiplication

Two matrices can be multiplied iff the number of columns of the 1st matrix [] is equal to the number of rows of the 2nd matrix [].

$$[A][B] = [A]_{m,p} [B]_{p,n} = [C]_{m,n}$$

note that the **p** dissapeared.

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Multiply Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 1(7) + 2(8) + 3(9) & 1(10) + 2(11) + 3(12) \\ 4(7) + 5(8) + 6(9) & 4(10) + 5(11) + 6(12) \end{bmatrix}$$

$$= \begin{bmatrix} 7 + 16 + 27 & 10 + 22 + 36 \\ 28 + 40 + 54 & 40 + 55 + 72 \end{bmatrix}$$

$$= \begin{bmatrix} 50 & 68 \\ 122 & 167 \end{bmatrix}$$

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Multiplication Continued

Note that the 1st row in [A] has the same number of elements as the 1st column in [B].

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Multiplication Continued

Also note that $[A]*[B]$ is not the same as $[B]*[A]$.

$$[B]_{3,2} [A]_{2,3} = [C]_{3,3}$$

Which is not the same as the output

$$\text{we got from } [A]_{2,3} [B]_{3,2} = [C]_{2,2}$$

In general, $[A][B] \neq [B][A]$

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Vector Products

$$[A]_{m,1} [B]_{1,n} = C_{mn}$$

col * Row = rectangular
vector vector matrix

$$[A]_{1,n} [B]_{n,1} = [C]_{1,1} = \text{Trivial Matrix or "Scalar"} = C$$

$$\begin{bmatrix} \text{row} \\ \text{vector} \end{bmatrix} \begin{bmatrix} \text{col} \\ \text{vector} \end{bmatrix} = \text{scalar}$$

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Vector Product Example

$$[C]_{3,2} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}_{3,1} \begin{bmatrix} 4 & 5 \end{bmatrix}_{1,2}$$

$$= \begin{bmatrix} 2(4) & 2(5) \\ -3(4) & -3(5) \\ 1(4) & 1(5) \end{bmatrix} = \begin{pmatrix} 8 & 10 \\ -12 & -15 \\ 4 & 5 \end{pmatrix}_{3,2}$$

Is $[B][A]$ possible? NO! The # of columns in $[B]$ isn't equal to the # of rows in $[A]$.

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Vector Product Example

$$\begin{bmatrix} 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ -4 \end{bmatrix} = [C]_{1,1} = C$$

$$= [4(6) + (-2)(1) + 3(-4)]$$

$$= [24 - 2 - 12] = [10] = C$$

There is no such thing as matrix division.
In order to achieve the same thing, we
must first invert one of the matrices 1st.

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Transpose of a Matrix

$$\begin{bmatrix} 1 & 6 & 9 \\ 5 & 4 & 1 \\ 7 & 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 7 \\ 6 & 4 & 3 \\ 9 & 1 & 8 \end{bmatrix}$$

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Determinant of a 2x2 matrix

$$D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3)$$

$$= 4 - 6 = -2$$

This was a cross product

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Expansion of Minors (or Cofactors)

For a determinant of order N

Minor = M_{ij} = minor corresponding to the i^{th} row and the j^{th} column.

It is a determinant of order $n-1$ obtained by crossing out the i^{th} row and the j^{th} column and then forming a new matrix.

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Cofactor

Cofactor - defined a slightly different way than a Minor.

$$\tilde{a}_{ij} = (-1)^{i+j} M_{ij}$$

It's obvious that for **even**

quantities of $i+j$, $(-1)^{i+j}$ is **positive**,
and for **odd** quantities

of $i+j$, $(-1)^{i+j}$ is **negative**.

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Cofactor Example

$$\det \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\tilde{a}_{1,1} M_{1,1} = (-1)^{1+1} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = (-1)^2 [(1)(1) - (3)(2)] = (1)[1 - 6] = -5$$

$$\tilde{a}_{1,2} M_{1,2} = (-1)^{1+2} \begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} = (-1)^3 [(-1)(1) - (3)(3)] = (-1)[-1 - 9] = 10$$

$$\tilde{a}_{1,3} M_{1,3} = (-1)^{1+3} \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = (-1)^4 [(-1)(2) - (1)(3)] = (1)[-2 - 3] = -5$$

$$\det[A] = a_{1,1} \tilde{a}_{1,1} + a_{1,2} \tilde{a}_{1,2} + a_{1,3} \tilde{a}_{1,3} = (1)(-5) + (2)(10) + (-1)(-5) \\ = -5 + 20 + 5 = \boxed{20}$$

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Matrix Inversion

In order to understand Matrix Inversion, we must understand a few more concepts first. Lets continue to calculate the remainder of the cofactor's for the matrix [A]. In doing so, we will no longer calculate the sign associated with the Minor.

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Matrix Inversion Continued

Remember how the sign was calculated. All we need to do is apply a (-1) if the (row + col) is odd and a (+1) if the (row + col) is even.

For example, the sign associated with the 1 in the 1st row, 1st column is (+) since 1+1 is even, while the sign associated with the 2 in the 1st row, 2nd column is (-) since 1+2 is odd.

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Example

$$[A] = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned} \tilde{a}_{1,1} &= + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5 & \tilde{a}_{1,2} &= - \begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} = 10 & \tilde{a}_{1,3} &= + \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = -5 \\ \tilde{a}_{2,1} &= - \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = -4 & \tilde{a}_{2,2} &= + \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 4 & \tilde{a}_{2,3} &= - \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 4 \\ \tilde{a}_{3,1} &= + \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 7 & \tilde{a}_{3,2} &= - \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = -2 & \tilde{a}_{3,3} &= + \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3 \end{aligned}$$

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Cofactor Matrix

$[\tilde{A}]$ = Matrix formed by replacing each element by it's cofactor.

$$[\tilde{A}] = \begin{bmatrix} -5 & 10 & -5 \\ -4 & 4 & 4 \\ 7 & -2 & 3 \end{bmatrix}$$

Adjoint Matrix

$$\text{adj}[A] = [\tilde{A}]^T = \begin{bmatrix} -5 & -4 & 7 \\ 10 & 4 & -2 \\ -5 & 4 & 3 \end{bmatrix}$$

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Identity Matrix

Identity Matrix = $[I]$

An Identity matrix is to matrix algebra as a 1 is to scalar algebra. The matrix **MUST BE SQUARE**.

$$[I] = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$[I][A] = [A] = [A][I]$ The dimensions of $[I]$ and $[A]$ must be the same.

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Inverse Matrix

The **Inverse matrix** is only defined for a **SQUARE matrix**. The inverse of a square matrix $[A]$ is denoted as and is defined as: $[A]^{-1}$

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

$$[A]^{-1} = \frac{\text{Adj}[A]}{\text{Det}[A]} = \frac{[\tilde{A}]^T}{|A|}$$

Be careful when using the $|A|$ to represent **Det** $[A]$. Don't get it mixed up with the Absolute value of A .

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Example (Slide 1)

$$\begin{aligned} 1x_1 + 2x_2 - 1x_3 &= -8 \\ -1x_1 + 1x_2 + 3x_3 &= 7 \\ 3x_1 + 2x_2 + 1x_3 &= 4 \end{aligned}$$

$[A]\vec{x} = [B]$ In this case $[B]$ is a vector, but we represent it as a Matrix.

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 7 \\ 4 \end{bmatrix}$$

We determined the Det $[A]$ earlier to be equal to 20.

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Example (Slide 2)

$$\text{Adj}[A] = \begin{bmatrix} -5 & -4 & 7 \\ 10 & 4 & -2 \\ -5 & 4 & 3 \end{bmatrix} \quad \text{Also determined this earlier.}$$

$$[A]^{-1} = \frac{\text{Adj}[A]}{|A|} = \frac{\begin{bmatrix} -5 & -4 & 7 \\ 10 & 4 & -2 \\ -5 & 4 & 3 \end{bmatrix}}{20} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{5} & \frac{7}{20} \\ \frac{1}{2} & \frac{1}{5} & -\frac{1}{10} \\ -\frac{1}{4} & \frac{1}{5} & \frac{3}{20} \end{bmatrix}$$

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Example (Slide 3)

$$\vec{x} = [A]^{-1}[B]$$

$$\vec{x} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{5} & \frac{7}{20} \\ \frac{1}{2} & \frac{1}{5} & -\frac{1}{10} \\ -\frac{1}{4} & \frac{1}{5} & \frac{3}{20} \end{bmatrix} \begin{bmatrix} -8 \\ 7 \\ 4 \end{bmatrix}$$

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Example (Slide 4)

$$\vec{x} = \begin{bmatrix} -\frac{1}{4}(-8) - \frac{1}{5}(7) + \frac{7}{20}(4) \\ \frac{1}{2}(-8) + \frac{1}{5}(7) - \frac{1}{10}(4) \\ -\frac{1}{4}(-8) + \frac{1}{5}(7) + \frac{3}{20}(4) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

So, $x_1 = 2$, $x_2 = -3$, and $x_3 = 4$

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Cramer's Rule (3x3 matrix only)

$$\begin{aligned} 1 &= 2x + 3y - 4z \\ 4 &= 3x - y - 2z \\ -7 &= 4x - 7y - 6z \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$$

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Cramer's Rule (The D matrix)

$$D = \begin{vmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{vmatrix} = + (2) \begin{vmatrix} -1 & -2 \\ -7 & -6 \end{vmatrix} - 3 \begin{vmatrix} 3 & -2 \\ 4 & -6 \end{vmatrix} + (-4) \begin{vmatrix} 3 & -1 \\ 4 & -7 \end{vmatrix}$$

$$D = + (2)[(-1)(-6) - (-2)(-7)] - 3[(3)(-6) - (-2)(4)] + (-4)[(3)(-7) - (-1)(4)]$$

$$D = 2[6 - 14] - 3[-18 - (-8)] + (-4)[-21 - (-4)]$$

$$D = 2[-8] - 3[-18 + 8] + (-4)[-21 + 4]$$

$$D = 2[-8] - 3[-10] + (-4)[-17] = -16 + 30 + 68 = \boxed{82}$$

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Cramer's Rule
(The D_x matrix)

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$$

$$D_x = \begin{bmatrix} 1 & 3 & -4 \\ 4 & -1 & -2 \\ -7 & -7 & -6 \end{bmatrix}$$

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Cramer's Rule

$$D_x = \begin{vmatrix} 1 & 3 & -4 \\ 4 & -1 & -2 \\ -7 & -7 & -6 \end{vmatrix} = 1 \begin{vmatrix} -1 & -2 \\ -7 & -6 \end{vmatrix} - 3 \begin{vmatrix} 4 & -2 \\ -7 & -6 \end{vmatrix} + (-4) \begin{vmatrix} 4 & -1 \\ -7 & -7 \end{vmatrix}$$

$$D_x = 1[(-1)(-6) - (-2)(-7)] - 3[(4)(-6) - (-2)(-7)] + (-4)[(4)(-7) - (-1)(-7)]$$

$$D_x = 1[6 - 14] - 3[-24 - 14] + (-4)[-28 - 7]$$

$$D_x = 1[-8] - 3[-38] - 4[-35]$$

$$D_x = -8 + 114 + 140 = 246$$

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Cramer's Rule
($x =$)

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 1 & 3 & -4 \\ 4 & -1 & -2 \\ -7 & -7 & -6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{vmatrix}} = \frac{246}{82} = 3$$

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Cramer's Rule
(**y** =)

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 2 & 1 & -4 \\ 3 & 4 & -2 \\ 4 & -7 & -6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{vmatrix}} = \frac{??}{82}$$

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Cramer's Rule
(**z** =)

$$z = \frac{D_z}{D} = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & -1 & 4 \\ 4 & -7 & -7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{vmatrix}} = \frac{??}{82}$$

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