# Fundamentals of Engineering Review

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**Math Review**

Linear Algebra

(Matrices)

# Matrix: A rectangular array of numbers or variables

$$[A] = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{bmatrix}$$

# Notations

m = # of rows
n = # of columns

**Other array notations are**

$$[A] = [A]_{m,n} = \left[a_{i,j}\right]$$

**Matrices obey all algebraic laws**

**<u>except</u> $AB \neq BA$**

# Example

$$[A] = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 4 & 6 \end{bmatrix} = [A]_{2,3}$$

$$[B] = \begin{bmatrix} 2 & 1 \\ 7 & -4 \\ 3 & 1 \end{bmatrix} = [B]_{3,2}$$

# Square Matrices

If m = n, the matrix is said to be "Square"

$$[C] = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 6 & 1 \\ -5 & 2 & 1 \end{bmatrix} = [C]_{3,3}$$

# Vector Dot Product

What is the angle between the vectors $F_1$ and $F_2$ ?

$$F_1 = 5\hat{i} + 4\hat{j} + 6\hat{k} \quad \text{and} \quad F_2 = 4\hat{i} + 10\hat{j} + 7\hat{k}$$

$$\cos\theta = \frac{F_1 \bullet F_2}{|F_1||F_2|}$$

$$= \frac{5(4) + 4(10) + 6(7)}{\sqrt{5^2 + 4^2 + 6^2}\sqrt{4^2 + 10^2 + 7^2}}$$

$$= \frac{20 + 40 + 42}{\sqrt{25 + 16 + 36}\sqrt{16 + 100 + 49}}$$

$$= \frac{102}{\sqrt{77}\sqrt{165}} = \frac{102}{112.7165} = .905$$

$$\cos^{-1}(.905) = \boxed{25.18^\circ}$$

# Column & Row Vectors

A matrix of ORDER $m \times 1$ is a column vector

Examples :

$$\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Column vector of ORDER 3

$$\bar{i} = \begin{bmatrix} i_1 & i_2 & i_3 & i_4 \end{bmatrix}$$

Row vector of ORDER 4

A matrix of ORDER $1 \times n$ is a row vector

# Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+2 \\ 3+3 & 4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

# Matrix Dot Product Example

$$4 \bullet \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 \bullet 2 & 4 \bullet 3 \\ 4 \bullet 4 & 4 \bullet 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 12 \\ 16 & 20 \end{bmatrix}$$

# Matrix Multiplication

Two matrices can be multiplied iff the number of columns of the 1st matrix [ ] is equal to the number of rows of the 2nd matrix [ ].

$$[A][B] = [A]_{m,p}\,[B]_{p,n} = [C]_{m,n}$$

note that the p dissapeared.

# Multiply Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 1(7)+2(8)+3(9) & 1(10)+2(11)+3(12) \\ 4(7)+5(8)+6(9) & 4(10)+5(11)+6(12) \end{bmatrix}$$

$$= \begin{bmatrix} 7+16+27 & 10+22+36 \\ 28+40+54 & 40+55+72 \end{bmatrix}$$

$$= \begin{bmatrix} 50 & 68 \\ 122 & 167 \end{bmatrix}$$

# Multiplication Continued

Note that the 1st row in [A] has the same number of elements as the 1st column in [B].

# Multiplication Continued

Also note that [A]*[B] is not the same as [B]*[A].

$$[B]_{3,2}\,[A]_{2,3} = [C]_{3,3}$$

Which is not the same as the output we got from $[A]_{2,3}\,[B]_{3,2} = [C]_{2,2}$

In general, $[A][B] \neq [B][A]$

# Vector Products

$[A]_{m,1}\,[B]_{1,n} = C_{mn}$

col vector * Row vector = rectangular matrix

$[A]_{1,n} \quad [B]_{n,1} = [C]_{1,1}$ = Trivial Matrix or "Scalar" = $C$

$$\begin{bmatrix}\text{row} \\ \text{vector}\end{bmatrix} \begin{bmatrix}\text{col} \\ \text{vector}\end{bmatrix} = \text{scalar}$$

# Vector Product Example

$$[C]_{3,2} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}_{3,1} \begin{bmatrix} 4 & 5 \end{bmatrix}_{1,2}$$

$$= \begin{bmatrix} 2(4) & 2(5) \\ -3(4) & -3(5) \\ 1(4) & 1(5) \end{bmatrix} = \begin{pmatrix} 8 & 10 \\ -12 & -15 \\ 4 & 5 \end{pmatrix}_{3,2}$$

Is [B] [A] possible? NO! The # of columns in [B] isn't equal to the # of rows in [A].

# Vector Product Example

$$\begin{bmatrix} 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ -4 \end{bmatrix} = [C]_{1,1} = C$$

$$= \left[4(6) + (-2)(1) + 3(-4)\right]$$

$$= \left[24 - 2 - 12\right] = [10] = C$$

There is no such thing as matrix division. In order to achieve the same thing, we must first invert one of the matrices 1st.

# Transpose of a Matrix

$$\begin{bmatrix} 1 & 6 & 9 \\ 5 & 4 & 1 \\ 7 & 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 7 \\ 6 & 4 & 3 \\ 9 & 1 & 8 \end{bmatrix}$$

# Determinant of a 2x2 matrix

$$D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3)$$

$$= 4 - 6 = \boxed{-2}$$

This was a cross product

# Expansion of Minors (or Cofactors)

For a determinant of order N

**Minor** = $M_{ij}$ = minor corresponding to the $i^{th}$ row and the $j^{th}$ column.

It is a determinant of order n-1 obtained by crossing out the $i^{th}$ row and the $j^{th}$ column and then and forming a new matrix.

# Cofactor

Cofactor - defined a slightly different way then a Minor.

$$\tilde{a}_{ij} = (-1)^{i+j} M_{ij}$$

It's obvious that for even quantities of i+j, $(-1)^{i+j}$ is positive, and for odd quantities of i+j, $(-1)^{i+j}$ is negative.

# Cofactor Example

$$\det\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\tilde{a}_{1,1}\ M_{1,1} = (-1)^{1+1}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = (-1)^2\left[(1)(1) - (3)(2)\right] = (1)\left[1 - 6\right] = -5$$

$$\tilde{a}_{1,2}\ M_{1,2} = (-1)^{1+2}\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} = (-1)^3\left[(-1)(1) - (3)(3)\right] = (-1)\left[-1 - 9\right] = 10$$

$$\tilde{a}_{1,3}\ M_{1,3} = (-1)^{1+3}\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = (-1)^4\left[(-1)(2) - (1)(3)\right] = (1)\left[-2 - 3\right] = -5$$

$$\det\left[A\right] = a_{1,1}\,\tilde{a}_{1,1} + a_{1,2}\,\tilde{a}_{1,2} + a_{1,3}\,\tilde{a}_{1,3} = (1)(-5) + (2)(10) + (-1)(-5)$$

$$= -5 + 20 + 5 = \boxed{20}$$

# Matrix Inversion

In order to understand Matrix Inversion, we must understand a few more concepts first. Lets continue to calculate the remainder of the cofactor's for the matrix [A]. In doing so, we will no longer calculate the sign associated with the Minor.

# Matrix Inversion Continued

Remember how the sign was calculated.
All we need to do is apply a (-1) if the (row + col) is odd and a (+1) if the (row + col) is even.

For example, the sign associated with the 1 in the 1st row, 1st column is (+) since 1+1 is even, while the sign associated with the 2 in the 1st row, 2nd column is (-) since 1+2 is odd.

# Example

$$[A] = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\tilde{a}_{1,1} = +\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5 \quad \tilde{a}_{1,2} = -\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} = 10 \quad \tilde{a}_{1,3} = +\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = -5$$

$$\tilde{a}_{2,1} = -\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = -4 \quad \tilde{a}_{2,2} = +\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 4 \quad \tilde{a}_{2,3} = -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 4$$

$$\tilde{a}_{3,1} = +\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 7 \quad \tilde{a}_{3,2} = -\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = -2 \quad \tilde{a}_{3,3} = +\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$$

# Cofactor Matrix

$$\left[\tilde{A}\right] = \text{Matrix formed by replacing each element by it's cofactor.}$$

$$\left[\tilde{A}\right] = \begin{bmatrix} -5 & 10 & -5 \\ -4 & 4 & 4 \\ 7 & -2 & 3 \end{bmatrix}$$

## Adjoint Matrix

$$\text{adj}[A] = [\tilde{A}]^T = \begin{bmatrix} -5 & -4 & 7 \\ 10 & 4 & -2 \\ -5 & 4 & 3 \end{bmatrix}$$

# Identity Matrix

## Identity Matrix = [I]

An Identity matrix is to matrix algebra as a 1 is to scalar algebra. The matrix MUST BE SQUARE.

$$[I] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$[I][A] = [A] = [A][I]$$

The demensions of [I] and [A] must be the same.

# Inverse Matrix

The Inverse matrix is only defined for a SQUARE matrix. The inverse of a square matrix [A] is denoted as and is defined as: $[A]^{-1}$

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

$$[A]^{-1} = \frac{Adj[A]}{Det[A]} = \frac{\left[\tilde{A}\right]^T}{|A|}$$

Be careful when using the $|A|$ to represent Det [A]. Don't get it mixed up with the Absolute value of A.

# Example (Slide 1)

$$\begin{aligned} 1x_1 &+ 2x_2 - 1x_3 = -8 \\ -1x_1 &+ 1x_2 + 3x_3 = 7 \\ 3x_1 &+ 2x_2 + 1x_3 = 4 \end{aligned}$$

$[A]\bar{x} = [B]$ In this case $[B]$ is a vector, but we represent it as a Matrix.

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -8 \\ 7 \\ 4 \end{bmatrix}$$

We determined the Det $[A]$ earlier to be equal to 20.

# Example (Slide 2)

$$\text{Adj}[A] = \begin{bmatrix} -5 & -4 & 7 \\ 10 & 4 & -2 \\ -5 & 4 & 3 \end{bmatrix}$$

Also determined this earlier.

$$[A]^{-1} = \frac{\text{Adj}[A]}{|A|} = \frac{\begin{bmatrix} -5 & -4 & 7 \\ 10 & 4 & -2 \\ -5 & 4 & 3 \end{bmatrix}}{20} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{5} & \frac{7}{20} \\ \frac{1}{2} & \frac{1}{5} & -\frac{1}{10} \\ -\frac{1}{4} & \frac{1}{5} & \frac{3}{20} \end{bmatrix}$$

# Example (Slide 3)

$$\bar{x} = [A]^{-1}[B]$$

$$\bar{x} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{5} & \frac{7}{20} \\ \frac{1}{2} & \frac{1}{5} & -\frac{1}{10} \\ -\frac{1}{4} & \frac{1}{5} & \frac{3}{20} \end{bmatrix} \begin{bmatrix} -8 \\ 7 \\ 4 \end{bmatrix}$$

# Example (Slide 4)

$$\bar{x} = \begin{bmatrix} -\frac{1}{4}(-8) - \frac{1}{5}(7) + \frac{7}{20}(4) \\ \frac{1}{2}(-8) + \frac{1}{5}(7) - \frac{1}{10}(4) \\ -\frac{1}{4}(-8) + \frac{1}{5}(7) + \frac{3}{20}(4) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

So, $x_1 = 2$, $x_2 = -3$, and $x_3 = 4$

# Cramer's Rule (3x3 matrix only)

$$1 = 2x + 3y - 4z$$

$$4 = 3x - y - 2z$$

$$-7 = 4x - 7y - 6z$$

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$$

# Cramer's Rule
# (The D matrix)

$$D = \begin{vmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{vmatrix} = +(2)\begin{vmatrix} -1 & -2 \\ -7 & -6 \end{vmatrix} - 3\begin{vmatrix} 3 & -2 \\ 4 & -6 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -1 \\ 4 & -7 \end{vmatrix}$$

$$D = +(2)\left[(-1)(-6) - (-2)(-7)\right] - 3\left[(3)(-6) - (-2)(4)\right] + (-4)\left[(3)(-7) - (-1)(4)\right]$$

$$D = 2[6 - 14] - 3[-18 - (-8)] + (-4)[-21 - (-4)]$$

$$D = 2[-8] - 3[-18 + 8] + (-4)[-21 + 4]$$

$$D = 2[-8] - 3[-10] + (-4)[-17] = -16 + 30 + 68 = \boxed{82}$$

# Cramer's Rule
# (The $D_x$ matrix)

$$\begin{bmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$$

$$D_x = \begin{bmatrix} 1 & 3 & -4 \\ 4 & -1 & -2 \\ -7 & -7 & -6 \end{bmatrix}$$

# Cramer's Rule

$$D_x = \begin{bmatrix} 1 & 3 & -4 \\ 4 & -1 & -2 \\ -7 & -7 & -6 \end{bmatrix} = 1\begin{vmatrix} -1 & -2 \\ -7 & -6 \end{vmatrix} - 3\begin{vmatrix} 4 & -2 \\ -7 & -6 \end{vmatrix} + (-4)\begin{vmatrix} 4 & -1 \\ -7 & -7 \end{vmatrix}$$

$$D_x = 1\left[(-1)(-6) - (-2)(-7)\right] - 3\left[(4)(-6) - (-2)(-7)\right] + (-4)\left[(4)(-7) - (-1)(-7)\right]$$

$$D_x = 1[6 - 14] - 3\left[(-24) - 14\right] + (-4)\left[(-28) - 7\right]$$

$$D_x = 1[-8] - 3[-38] - 4[-35]$$

$$D_x = -8 + 114 + 140 = \boxed{246}$$

# Cramer's Rule
## (x = )

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 1 & 3 & -4 \\ 4 & -1 & -2 \\ -7 & -7 & -6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{vmatrix}} = \frac{246}{82} = \boxed{3}$$

# Cramer's Rule
(y= )

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 2 & 1 & -4 \\ 3 & 4 & -2 \\ 4 & -7 & -6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{vmatrix}} = \frac{??}{82}$$

# Cramer's Rule
## (z = )

$$z = \frac{D_z}{D} = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & -1 & 4 \\ 4 & -7 & -7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{vmatrix}} = \frac{??}{82}$$