Fundamentals of Engineering Review

·Richard L. Jones
·Math Review
·Differential Calculus

Derivatives

- Define derivatives
- Rules for finding derivatives
- Applications for derivatives
  - Local extrema
  - Rates of change
  - Approximating change

Definition

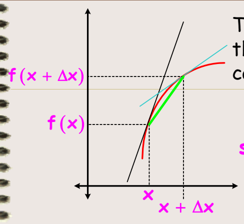
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x \to \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = D_x(f(x)) = \frac{df(x)}{dx}$$

means derivative with respect to x

Interpreted as:

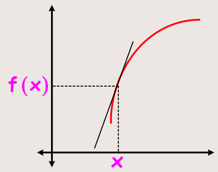
- Slope of a tangent line
- Rate of change of the value of a function



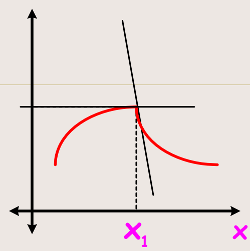
The green line which connects the two points is sometimes called the Secant Line

$$\text{slope} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If you make the points closer and closer together by decreasing the size of delta x, you get the derivative or slope at that point.



8/24/2010 4



When a point has a different slope direction
On each side of it, $f'(x)$ does not exist at that point.

8/24/2010 5

Finding derivatives

⇒ **Linearity**

$d_x(a \cdot F(x) + b \cdot G(x)) = a \cdot d_x(F(x)) + b \cdot d_x(G(x))$

⇒ **Power rule**

$D_x(x^r) = rx^{r-1}$

Example

$D_x(x^3) = 3x^2$

$D_x(\sin x) = \cos x \qquad D_x e^x = e^x$

$D_x(\cos x) = -\sin x \qquad D_x(\ln x) = \frac{1}{x}$

8/24/2010 6

Finding derivatives cont)

⇒ Product rule

$D_x(f(x) \bullet g(x)) = f'(x)g(x) + f(x)g'(x)$

e.g. $D_x(x^2 \sin x) = 2x(\sin x) + x^2(\cos x)$

⇒ Quotient rule

$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

e.g. $D_x\left(\frac{x^2}{\sin x}\right) = \frac{2x\sin x - x^2\cos x}{(\sin x)^2}$

8/24/2010 7

Finding derivatives cont)

⇒ Chain rule: Used for composite functions

$D_x(f(g(x))) = f'(g(x) \bullet g'(x))$

assign some variable names

$u = g(x) \Rightarrow \frac{df(u)}{dx} = \frac{df'(u)}{du} \bullet \frac{du}{dx}$

e.g. $D_x(\sin x)^3 = 3(\sin x)^2 \cos x$

8/24/2010 8

Combining some rules

$D_x\left(\sqrt{x^2+2x+1}\right) = D_x(x^2+2x+1)^{\frac{1}{2}}$

$= \frac{1}{2}(x^2+2x+1)^{\frac{-1}{2}}(2x+2)$

$= \boxed{\frac{(x+1)}{(x^2+2x+1)^{1/2}}}$

8/24/2010 9

Combining rules cont)

$$D_x\left[(\sin 2x)\bullet\ln\left(x^2+1\right)\right] = D_x(\sin 2x)\bullet\ln\left(x^2+1\right) + \sin 2x\bullet D_x\left[\ln\left(x^2+1\right)\right]$$

$$= \boxed{\begin{array}{l}(2)\cos 2x\bullet\ln\left(x^2+1\right)\\ +\sin 2x\left[\dfrac{1}{x^2+1}\right](2x)\end{array}}$$

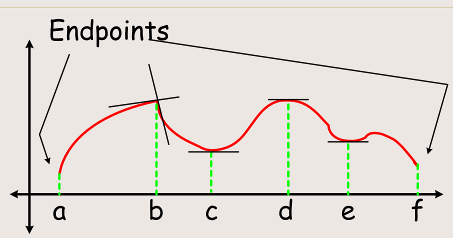
8/24/2010 10

One more example

$$D_x(\ln\pi) = \frac{1}{\pi}\bullet D_x(\pi) = \frac{1}{\pi}\bullet 0 = 0$$

8/24/2010 11

Extrema

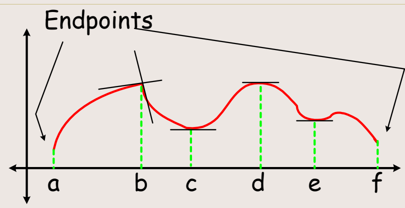


One application is when the maxima or minima points of a function are desired. These points are the extrema a function.

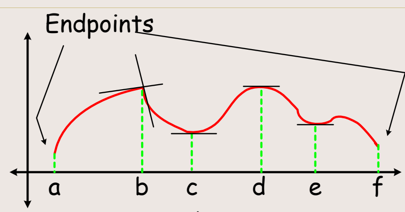
8/24/2010 12

To find the maxima or minima

- **Find the critical numbers**
- **Decide what happens there**
- **Check the endpoints**

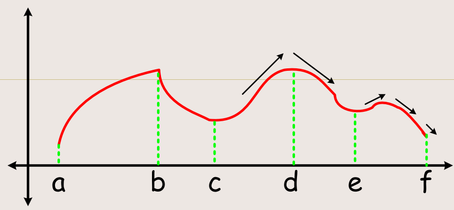


One way of finding the maximum or minimum points is to find the HORIZONTAL tangent locations (not including the endpoints)



Maxima or minima occur where:
$f'(x)=0$ or $f'(x)$ does not exist, as well as at the endpoints where $f'(x)$ is not possible.

These points are called the "Critical numbers"



+ ⇒ − ⇒ maxima

- ⇒ + ⇒ minima

Levels off but keeps → rising

or → falling ⇒ neither

disadvantage of this test requires f'(x) on each side of a critical point.

8/24/2010 16

1st derivative test

f'(x) > 0

means that the value is increasing

f'(x) < 0

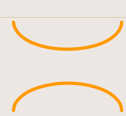
means that the value is decreasing

8/24/2010 17

2nd derivative test

f"(x) > 0 ⇒ concave up

f"(x) < 0 ⇒ concave down



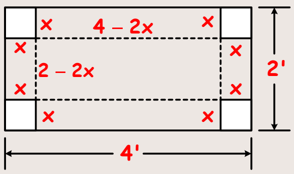
f"(x) = 0 ⇒ no conclusion

Minimum Maximum No info

8/24/2010 18

Derivative problem #1

An open box is to be created by cutting a square out of each corner of a 2 x 4 ft sheet of cardboard and folding it up into the box.
What size square should be cut out to maximize the volume of the box?



Problem # 1 cont)

$\text{Vol} = L \bullet w \bullet h$

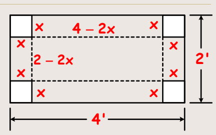
$= x(4 - 2x)(2 - 2x)$

$= 4x^3 - 12x^2 + 8x$

Critical numbers are

$V'(x) = 12x^2 - 24x + 8 = 0$

$x = 1.577'$ or $x = 0.423'$



$V''(x) = 24x - 24$

$V''(1.577) = 13.85$ Minimum

$V''(0.423) = -13.85$ Maximum!!

Cut 0.423 ft square out of each corner

Other uses of derivatives

- Rate of change
- D_t(position (t)) = Velocity (t)
- D_t (Velocity (t)) = Acceleration (t)

Related rates

We know the rate of change of one of the variables.

$y = f(x)$ x and y vary with t

$$\frac{dy}{dt} = \frac{df(x)}{dt} = \frac{df(x)}{dx} \bullet \frac{dx}{dt}$$

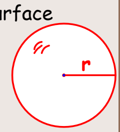
chain rule

$$= \frac{dy}{dt} = \frac{dy}{dx} \bullet \frac{dx}{dt} \quad \left(\text{we hopefully know } \frac{dx}{dt}\right)$$

8/24/2010 22

Application Example

A spherical balloon is inflating so that its radius is increasing 1" per min. How fast is the surface area increasing when the radius is 30" ?



surface area $= 4\pi r^2 = A$

$$\frac{dr}{dt} = 1''/\text{min} \quad \text{and} \quad r = 30''$$

$$\frac{dA}{dt} = \frac{dA}{dr} \bullet \frac{dr}{dt} = \frac{d(4\pi r^2)}{dr} \bullet (1''/\text{min})$$

$$= 8\pi r \bullet (1''/\text{min}) = 8\pi(30'') \bullet (1''/\text{min})$$

$$= \boxed{754 \ \text{in}^2/\text{min}}$$

8/24/2010 23

Using differentials to approximate answers

Use differentials to approximate ln (1.01)

$f(x) = \ln(1)$

$\ln(1) = 0$ so let $x_0 = 1$

$x_1 = 1.01$ so $\Delta x = 0.01$

Δx is small compared to the whole so result should be fairly accurate.

$$f'(x) = \frac{1}{x} \quad f'(x_0) = 1 \quad \left(\frac{1}{1} = 1\right)$$

$$f(x_1) \approx f(x_0) + f'(x_0) \cdot \Delta x$$

$$\approx 0 + 1 \cdot (0.01) = .01$$

check : $\ln(1.01) = 0.00995 \Rightarrow$ error $= \frac{1}{2}\%$

8/24/2010 24

Example

Given : $y(x) = 3x^3 - 2x^2 + 7$

What is the slope of the function at $x = 4$?

$y'(x) = 3(3)x^2 - 2(2)x = 9x^2 - 4x$

$y'(4) = 9(4)^2 - 4(4)$

$= 9(16) - 8 = 144 - 8$

$= \boxed{136}$

8/24/2010 25

Example

What is the maximum of the function

$y = -x^3 + 3x$ for $x \geq -1$

$y' = -3x^2 + 3$ and $y'' = -6x$

When $y' = 0 = -3x^2 + 3 \Rightarrow x^2 = 1 \therefore x = \pm 1$

$y''(1) = -6(1) = -6 < 0 \quad \therefore$ a maximum

$y''(-1) = -6(-1) = 6 > 0 \quad \therefore$ a minimum

so, $y(-1) = (-1)^3 + 3 = -1 + 3 = 2$(max)

8/24/2010 26

Example (cont)

What is the point of inflection of the function

$y = -x^3 + 3x - 2$

$y' = -3x^2 + 3$ and $y'' = -6x$

$y = f(x)$ is an inflection point for $x = a$ where

$f''(a) = 0$ and $f''(a)$ changes sign about $x = a$, so,

$y''(0)$ when $x = 0$ and $y'' > 0$ for $x < 0$

and $y'' < 0$ for $x > 0$

Therefore, this is an inflection point

$y(0) = -(0)^3 + 3(0) - 2 = -2$ so the answer is $(0,-2)$

8/24/2010 27