Fundamentals of Engineering Review

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- Math Review
- Differential Calculus

Derivatives

- Define derivatives
- Rules for finding derivatives
- Applications for derivatives
  - Local extrema
  - Rates of change
  - Approximating change

Definition

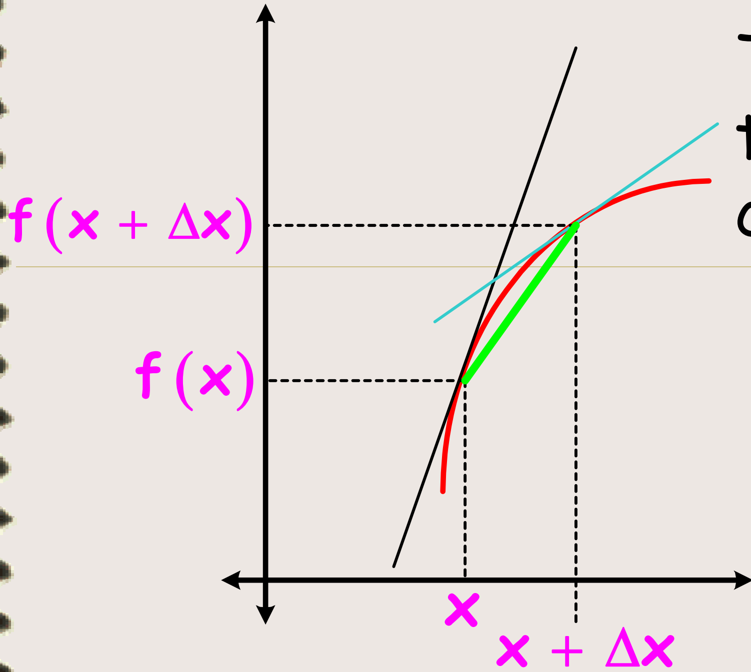
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x \to \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = D_x(f(x)) = \frac{df(x)}{dx}$$

means derivative with respect to x

Interpreted as:

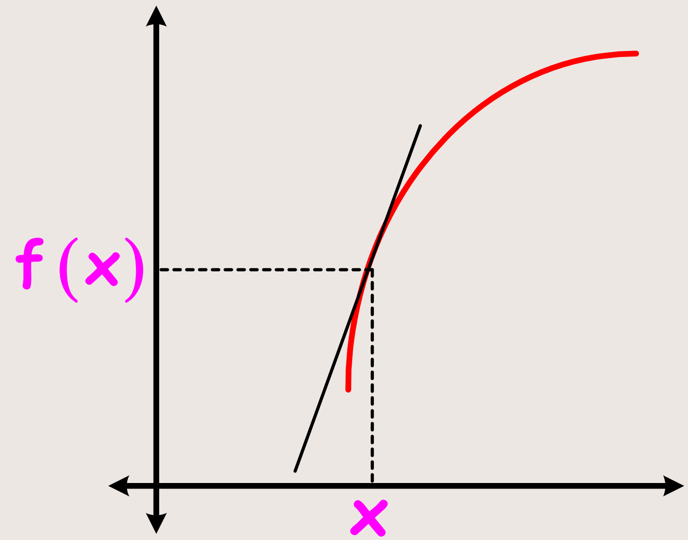
- **Slope of a tangent line**
- **Rate of change of the value of a function**

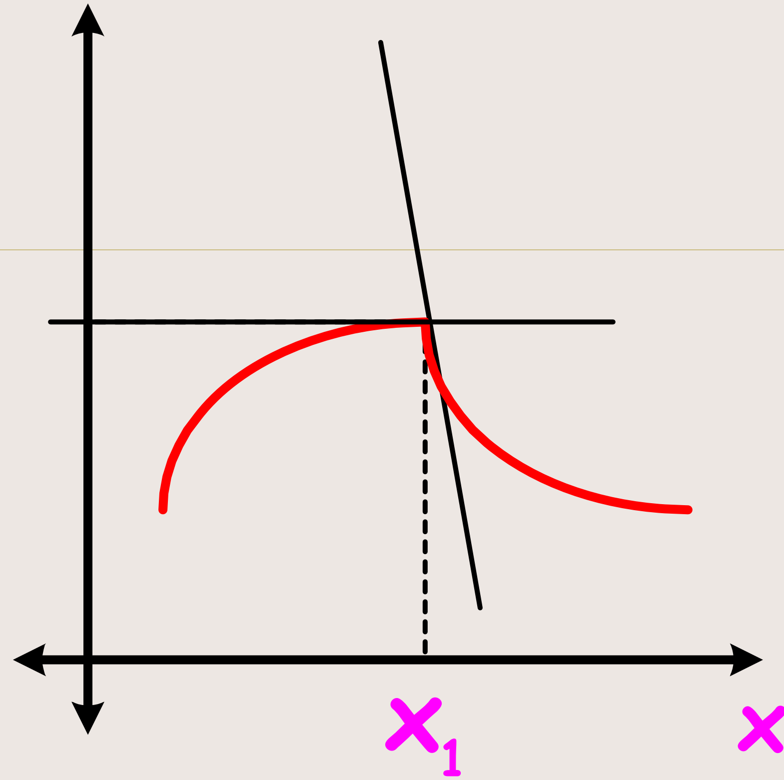


The green line which connects the two points is sometimes called the Secant Line

$$\text{slope} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If you make the points closer and closer together by decreasing the size of delta x, you get the derivative or slope at that point.





When a point has a different slope direction
On each side of it, f′(x) does not exist at that point.

Finding derivatives

⇒ **Linearity**

$$d_x\left(a \cdot F(x) + b \cdot G(x)\right) = a \cdot d_x\left(F(x)\right) + b \cdot d_x\left(G(x)\right)$$

⇒ **Power rule**

$$D_x\left(x^r\right) = r x^{r-1}$$

Example

$$D_x\left(x^3\right) = 3x^2$$

$$D_x(\sin x) = \cos x \qquad D_x e^x = e^x$$

$$D_x(\cos x) = -\sin x \qquad D_x(\ln x) = \frac{1}{x}$$

Finding derivatives cont)

⇒ **Product rule**

$$D_x\left(f(x)\bullet g(x)\right) = f'(x)\,g(x) + f(x)\,g'(x)$$

e.g. $$D_x\left(x^2 \sin x\right) = 2x(\sin x) + x^2(\cos x)$$

⇒ **Quotient rule**

$$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

e.g. $$D_x\left(\frac{x^2}{\sin x}\right) = \frac{2x\sin x - x^2\cos x}{(\sin x)^2}$$

Finding derivatives cont)

$\Rightarrow$ **Chain rule: Used for composite functions**

$$D_x\left(f\left(g\left(x\right)\right)\right) = f'\left(g\left(x\right) \bullet g'\left(x\right)\right)$$

assign some variable names

$$u = g(x) \Rightarrow \frac{df\left(u\right)}{dx} = \frac{df'\left(u\right)}{du} \bullet \frac{du}{dx}$$

e.g. $D_x\left(\sin x\right)^3 = 3\left(\sin x\right)^2 \cos x$

Combining some rules

$$D_x\left(\sqrt{x^2+2x+1}\right)=D_x\left(x^2+2x+1\right)^{\frac{1}{2}}$$

$$=\frac{1}{2}\left(x^2+2x+1\right)^{\frac{-1}{2}}(2x+2)$$

$$=\boxed{\frac{(x+1)}{\left(x^2+2x+1\right)^{1/2}}}$$

Combining rules cont)

$$D_x\left[(\sin 2x)\bullet\ln\left(x^2+1\right)\right]=D_x(\sin 2x)\bullet\ln\left(x^2+1\right)$$
$$+\sin 2x\bullet D_x\left[\ln\left(x^2+1\right)\right]$$
$$=\boxed{\begin{array}{l}(2)\cos 2x\bullet\ln\left(x^2+1\right)\\ +\sin 2x\left[\dfrac{1}{x^2+1}\right](2x)\end{array}}$$

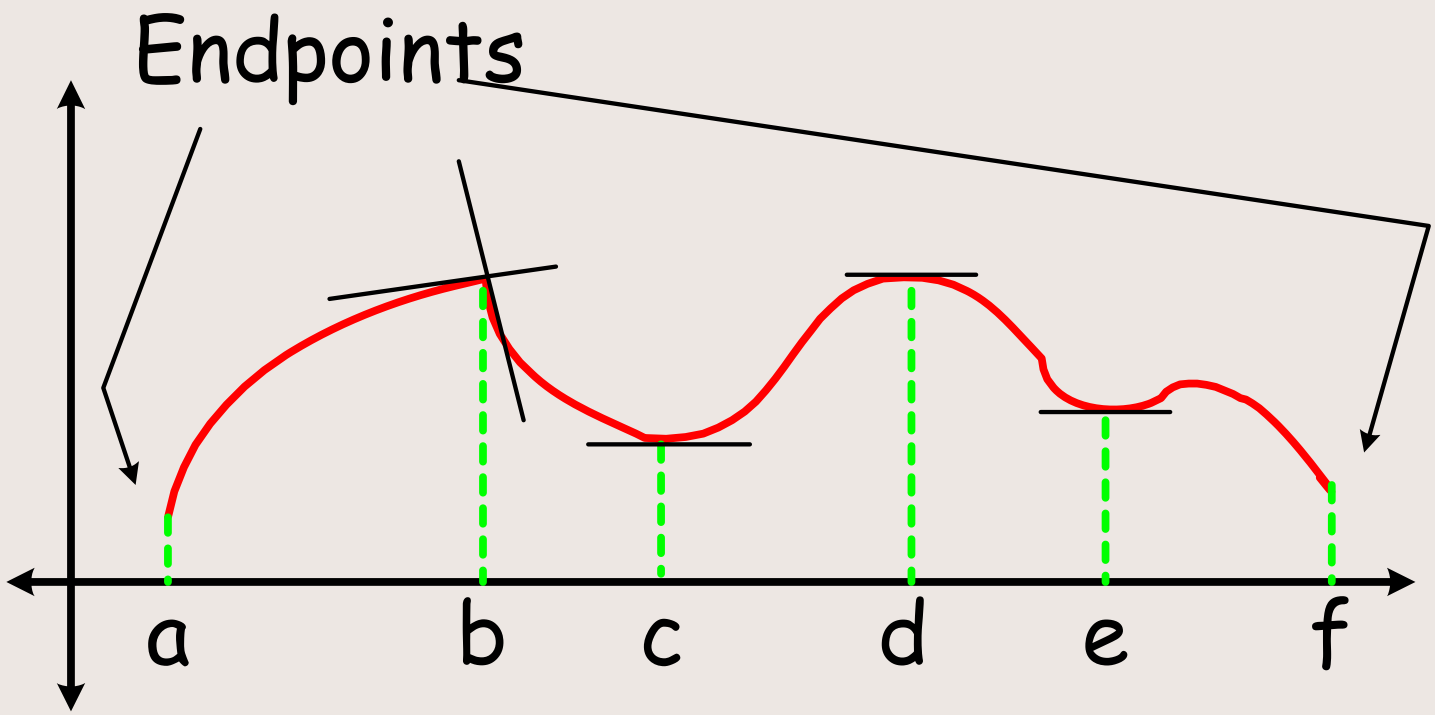
One more example

$$D_x(\ln \pi) = \frac{1}{\pi} \bullet D_x(\pi)$$

$$= \frac{1}{\pi} \bullet 0$$

$$= 0$$

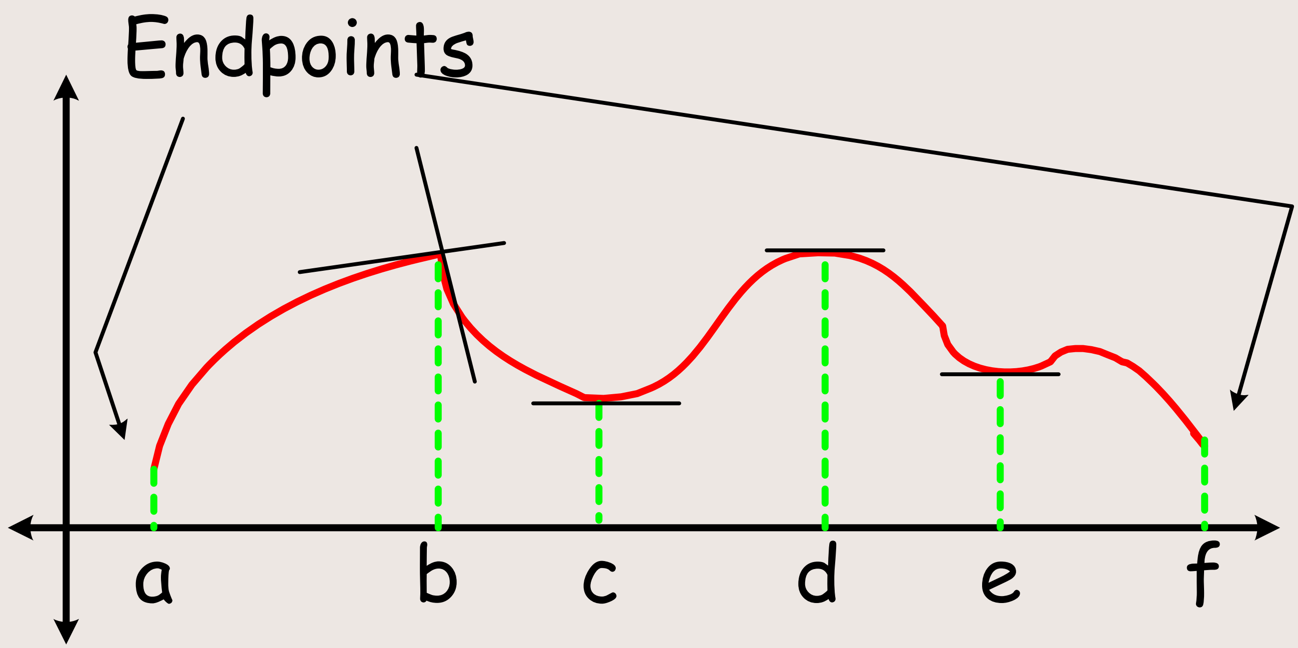
Extrema



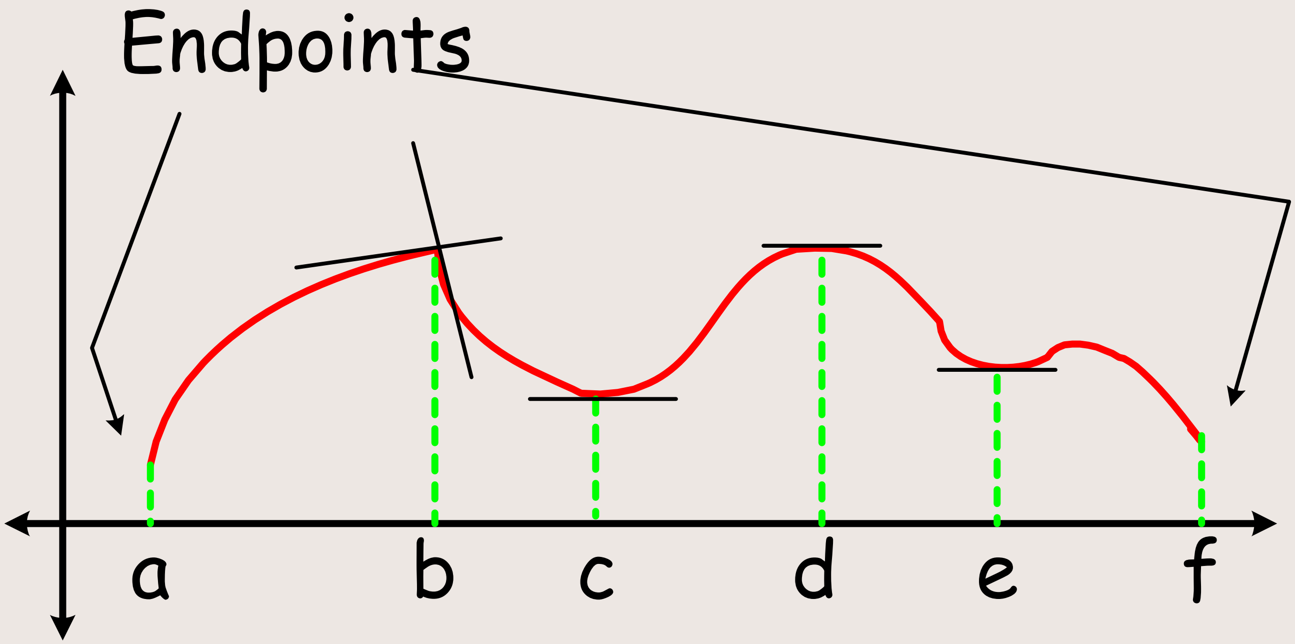
One application is when the maxima or minima points of a function are desired. These points are the extrema a function.

To find the maxima or minima

- **Find the critical numbers**
- **Decide what happens there**
- **Check the endpoints**



One way of finding the maximum or minimum points is to find the HORIZONTAL tangent locations (not including the endpoints)



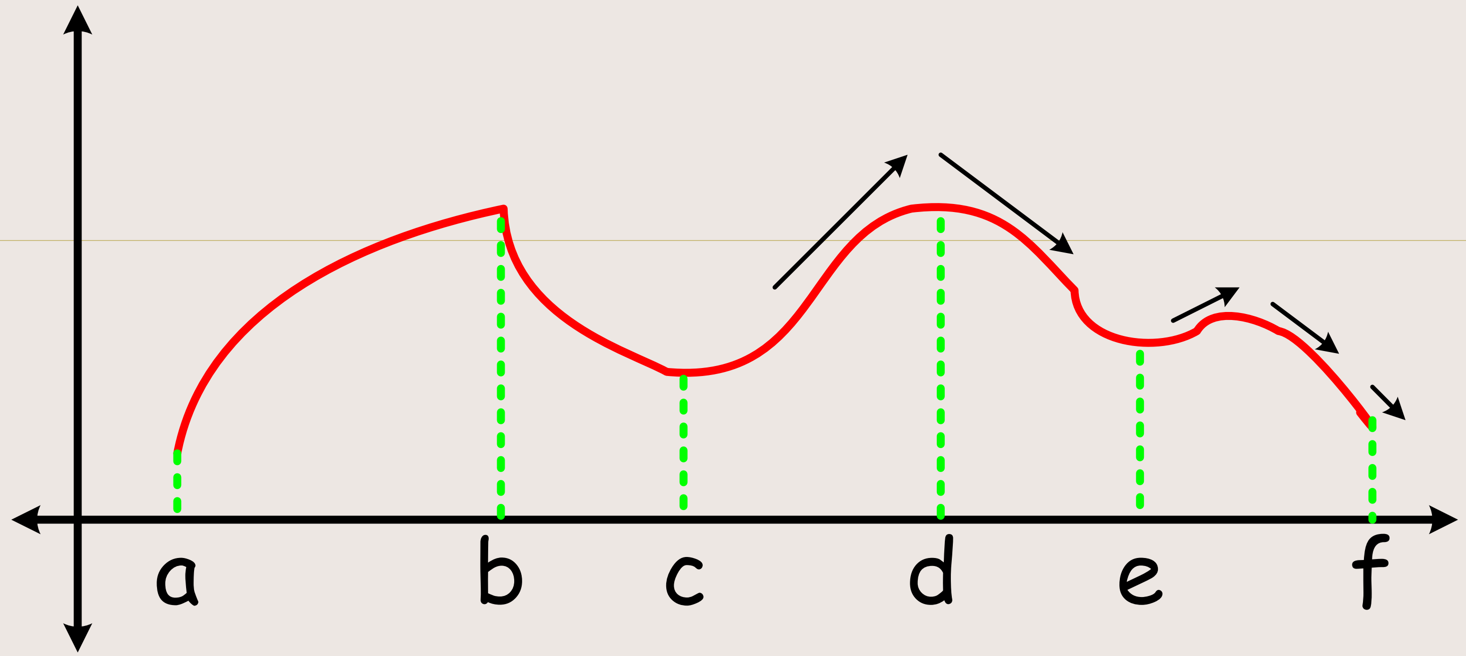
Maxima or minima occur where:

$f'(x)=0$ or $f'(x)$ does not exist,

as well as at the endpoints where

$f'(x)$ is not possible.

These points are called the "Critical numbers"



$+ \Rightarrow -$ $\Rightarrow$ **maxima**

$- \Rightarrow +$ $\Rightarrow$ **minima**

Levels off but keeps $\rightarrow$ **rising**

or $\rightarrow$ **falling** $\Rightarrow$ **neither**

disadvantage of this test requires $f'(x)$

on each side of a critical point.

1^{st} derivative test

$f'(x) > 0$

means that the value is increasing

$f'(x) < 0$

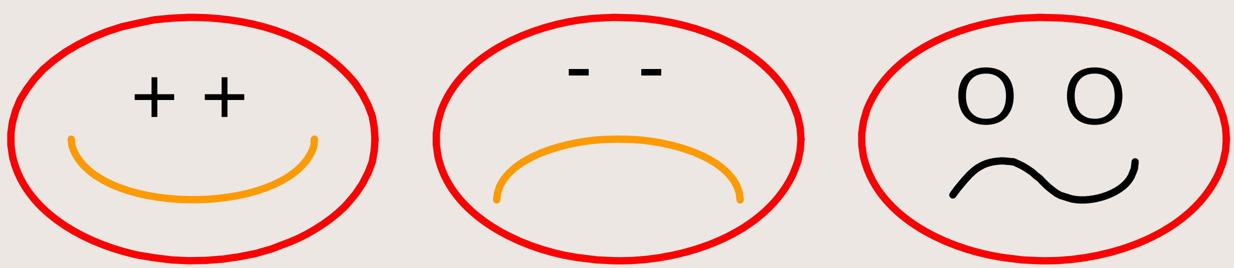
means that the value is decreasing

2^{nd} derivative test

$f''(x) > 0 \Rightarrow$ concave up

$f''(x) < 0 \Rightarrow$ concave down

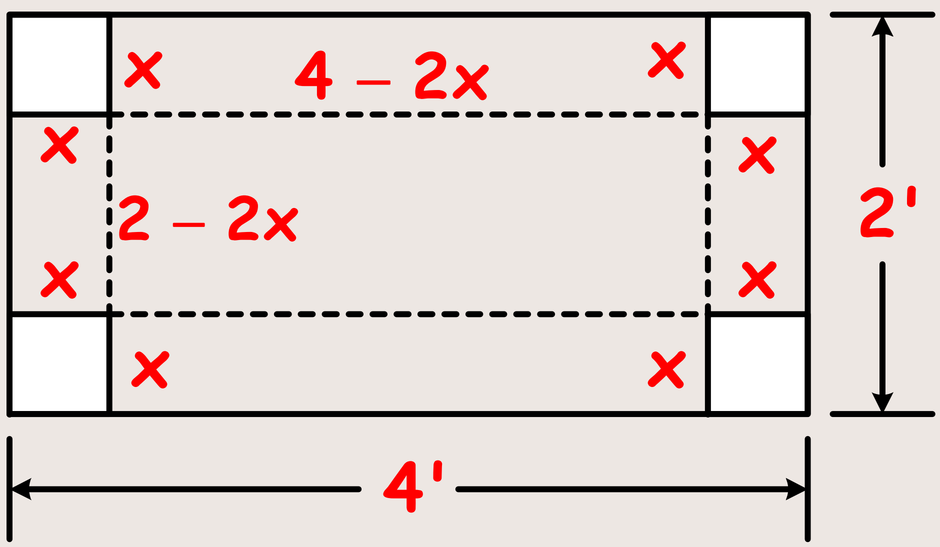
$f''(x) = 0 \Rightarrow$ no conclusion



Minimum **Maximum** **No info**

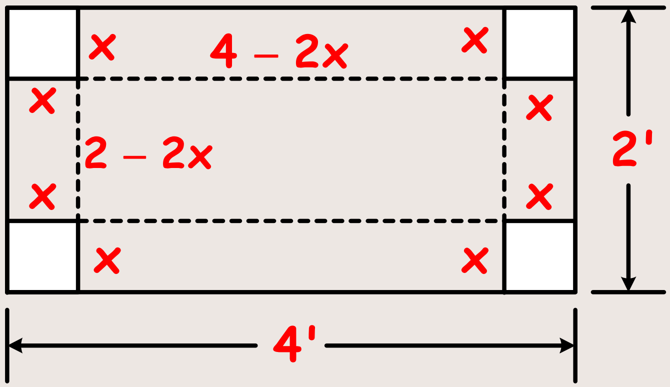
Derivative problem #1

An open box is to be created by cutting a square out of each corner of a **2 x 4 ft** sheet of cardboard and folding it up into the box. **What size square should be cut out to maximize the volume of the box?**



Problem # 1 cont)

$$Vol = L \bullet w \bullet h$$
$$= x(4-2x)(2-2x)$$
$$= 4x^3 - 12x^2 + 8x$$



Critical numbers are

$$V'(x) = 12x^2 - 24x + 8 = 0$$

$x = 1.577'$ or $x = 0.423'$

$$V''(x) = 24x - 24$$

$$V''(1.577) = 13.85$$

Minimum

$$V''(0.423) = -13.85$$

Maximum!!

Cut 0.423 ft square out of each corner

Other uses of derivatives

- Rate of change
- D_t(position (t)) = Velocity (t)
- D_t (Velocity (t)) = Acceleration (t)

Related rates

We know the rate of change of one of the variables.

$y = f(x)$ x and y vary with t

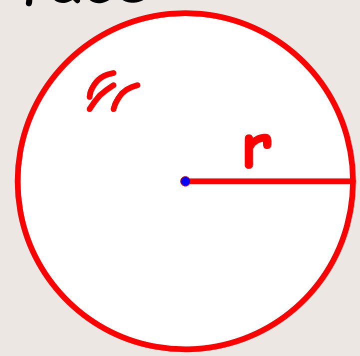
$$\frac{dy}{dt} = \frac{df(x)}{dt} = \frac{df(x)}{dx} \bullet \frac{dx}{dt}$$

chain rule

$$= \frac{dy}{dt} = \frac{dy}{dx} \bullet \frac{dx}{dt} \quad \left(\text{we hopefully know } \frac{dx}{dt}\right)$$

Application Example

A spherical balloon is inflating so that its radius is increasing 1" per min. How fast is the surface area increasing when the radius is 30" ?



surface area $= 4\pi r^2 = A$

$$\frac{dr}{dt} = 1''/\text{min} \quad \text{and} \quad r = 30''$$

$$\frac{dA}{dt} = \frac{dA}{dr} \bullet \frac{dr}{dt} = \frac{d\left(4\pi r^2\right)}{dr} \bullet \left(1''/\text{min}\right)$$

$$= 8\pi r \bullet \left(1''/\text{min}\right) = 8\pi\left(30''\right) \bullet \left(1''/\text{min}\right)$$

$$= \boxed{754\ \text{in}^2/\text{min}}$$

Using differentials to approximate answers

Use differentials to approximate ln (1.01)

$$f(x) = \ln(1)$$

$\ln(1) = 0$ so let $x_0 = 1$

$x_1 = 1.01$ so $\Delta x = 0.01$

Δx is small compared to the whole

so result should be fairly accurate.

$$f'(x) = \frac{1}{x} \qquad f'(x_0) = 1 \qquad \left(\frac{1}{1} = 1\right)$$

$$f(x_1) \approx f(x_0) + f'(x_0) \cdot \Delta x$$

$$\approx 0 + 1 \cdot (0.01) = .01$$

check : $\ln(1.01) = 0.00995 \Rightarrow$ error $= \frac{1}{2}\%$

Example

Given : $y(x) = 3x^3 - 2x^2 + 7$

What is the slope of the function at $x = 4$?

$$y'(x) = 3(3)x^2 - 2(2)x = 9x^2 - 4x$$

$$y'(4) = 9(4)^2 - 4(4)$$

$$= 9(16) - 8 = 144 - 8$$

$$= \boxed{136}$$

Example

What is the maximum of the function

$y = -x^3 + 3x \quad \text{for} \quad x \geq -1$

$y' = -3x^2 + 3 \quad \text{and} \quad y'' = -6x$

When $y' = 0 = -3x^2 + 3 \Rightarrow x^2 = 1 \therefore x = \pm 1$

$y''(1) = -6(1) = -6 < 0 \quad \therefore$ a maximum

$y''(-1) = -6(-1) = 6 > 0 \quad \therefore$ a minimum

so, $y(-1) = (-1)^3 + 3 = -1 + 3 = 2(\text{max})$

Example (cont)

What is the point of inflection of the function

$y = -x^3 + 3x - 2$

$y' = -3x^2 + 3$ and $y'' = -6x$

$y = f(x)$ is an inflection point for $x = a$ where

$f''(a) = 0$ and $f''(a)$ changes sign about $x = a$, so,

$y''(0)$ when $x = 0$ and $y'' > 0$ for $x < 0$

and $y'' < 0$ for $x > 0$

Therefore, this is an inflection point

$y(0) = -(0)^3 + 3(0) - 2 = -2$ so the answer is $(0,-2)$