Integrals

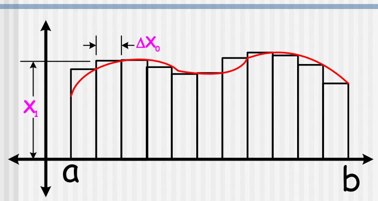
- Define Integrals
- Techniques of Integration
- Applications
  - Area
  - Work

Quote

You never really understand mathematics, you just get used to it.

John von Neumann
Famous mathematician

Integrals defined



a and b are the boundaries

$\Delta x \equiv$ width

$x_i \equiv$ height

$\|P\| \equiv$ Norm of the Partition $=$ largest Δx

$$\text{Area} = \lim_{\|P\| \to 0} \left(\sum_{i=0}^{n} f(x)\, \Delta x_i \right) = \int_a^b f(x)\, dx$$

Fundamental Theorem of Calculus

$D_x[F(x)] = f(x)$ $\quad$ $F(x) \equiv$ anti – derivative

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$\int f(x)\,dx = F(x) + C$ $\quad$ $\int \equiv$ Indefinite integral

$\Uparrow$

find the anti-derivative

Rules

$$\int u^r du = \frac{u^{r+1}}{r+1} + C \qquad \text{if} \quad r \neq -1$$

$$\int u^{-1} du = \int \frac{du}{u} = \ln|u| + C$$

$$\int \sin u \; du = -\cos u + C$$

$$\int \cos u \; du = \sin u + C$$

$$\int e^u du = e^u + C$$

Integration Methods

$$\int [af(u) + bg(u)]du = a\int f(u)\,du + b\int g(u)\,du$$

$$\int f(u)\,du$$

Must match exactly

u substitution

$$\int \left(f\left[g(x)\right] \bullet g'(x) \right) dx$$

let $u = g(x)$ $\quad \therefore \frac{du}{dx} = g'(x)$

and $\quad du = g'(x)dx$

$$\Rightarrow \int f(u) du$$

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u substitution example

$$\int \left(\sin x^3 \right) x^2 dx$$

let $u=x^3$ so $du=3x^2dx$

This puts a 3 into the equation so we will have to multiply by $\frac{1}{3}$ to take it out

$$\frac{1}{3}\int (\sin u)\, du = \frac{-1}{3}\cos u + C = \boxed{\frac{-1}{3}\cos x^3 + C}$$

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Another 'u' substitution Example

$$\int \frac{(\ln x)^3}{x} dx$$

let $u = \ln x$ $\quad \therefore \quad du = \frac{1}{x}dx$

$$\int u^3 du = \frac{1}{4}u^4 + C = \boxed{\frac{1}{4}(\ln x)^4 + C}$$

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Still another 'u' substitution example

Find $\int (e^x + 2x)^2 (e^x + 2)\,dx$

let $u(x) = e^x + 2x$

so $du = (e^x + 2)\,dx$

$$= \int (e^x + 2x)^2 (e^x + 2)\,dx$$

$$= \int u^2 du = \frac{u^3}{3} + C$$

$$= \boxed{\frac{1}{3}(e^x + 2x)^3 + C}$$

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Integration by parts

$$\boxed{\int udv = u \cdot v - \int vdu}$$

needs u, v, and du

Example Solve $\int x \sin x dx$

let $\begin{cases} u = x & dv = \sin x dx \\ du = dx & v = -\cos x \end{cases}$

$$= -x\cos x - \int -\cos x dx = -x\cos x + \int \cos x dx$$

$$= \boxed{-x\cos x + \sin x + C}$$

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Another "by parts" example

$\int x^2 \ln x dx \Rightarrow$ let $\boxed{\begin{matrix} u = \ln x & dv = x^2 dx \\ du = \frac{1}{x}dx & v = \frac{1}{3}x^3 \end{matrix}}$

$uv - \int vdu$

$$\int x^2 \ln x dx = \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3\right)\left(\frac{1}{x}dx\right)$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$

$$= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$$

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Still another "By parts" example

Find $\int x^2 e^x dx$

Let $u = x^2 \quad dv = e^x dx$; $du = 2x \quad v = e^x$

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$
$$= x^2 e^x - 2\int x e^x dx$$

From handbook $\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1)$, so

$$\int x^2 e^x dx = x^2 e^x - 2e^x (x - 1) + C$$
$$= x^2 e^x - 2e^x (xe^x - e^x) + C$$

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Partial Fraction Expansion

$$\int \frac{P(x)}{Q(x)} dx$$

$P(x)$ and $Q(x)$ are polynomials with an order of $P(x) < Q(x)$

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Partial Fraction Expansion Example

Solve $\int \frac{6x^2 + 11x - 1}{x^3 - 3x - 2} dx$

The denominator factors to $\Rightarrow (x+1)^2 (x - 2)$

$$\frac{6x^2 + 11x - 1}{x^3 - 3x - 2} = \frac{A}{(x-2)} + \frac{C_2}{x+1} + \frac{C_1}{(x+1)^2}$$

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Example Continued

$$\frac{6x^2+11x-1}{(x+1)^2(x-2)} = \frac{A}{(x-2)} + \frac{C_2}{x+1} + \frac{C_1}{(x+1)^2}$$

$$A = \left.\frac{6x^2+11x-1}{\cancel{(x-2)}(x+1)^2}\right|_{x=2} = \frac{6(2)^2+11(2)-1}{(2+1)^2}$$

$$= \frac{6(4)+22-1}{3^2}$$

$$= \frac{24+22-1}{3^2} = \frac{45}{9} = \boxed{5}$$

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Partial Fraction Expansion Multiple Order Poles

$$F(x) = \frac{Q(x)}{(x+a)^i}$$

$$C_k = \left.\frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}Q(x)\right|_{x=-a}$$

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Multiple Order Poles Example

$$\frac{6x^2+11x-1}{(x+1)^2(x-2)} = \frac{5}{(x-2)} + \frac{C_2}{x+1} + \frac{C_1}{(x+1)^2}$$

$$\boxed{F(x) = \frac{Q(x)}{(x+a)^i} \Rightarrow C_k = \left.\frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}Q(x)\right|_{x=-a}}$$

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Example Continued

$$C_1 = \frac{1}{(k-1)!}\left(\frac{d^{k-1}}{dx^{k-1}}\frac{6x^2+11x-1}{x-2}\right)\Bigg|_{x=-1} \qquad (k=1)$$

$$= \frac{1}{(1-1)!}\left(\frac{d^{1-1}}{dx^{1-1}}\frac{6x^2+11x-1}{x-2}\right)\Bigg|_{x=-1}$$

$$= (1)\left(\frac{d^0}{dx^0}\frac{6x^2+11x-1}{x-2}\right)\Bigg| = \left(\frac{6x^2+11x-1}{x-2}\right)\Bigg|_{x=-1}$$

$$= \frac{6(-1)^2+11(-1)-1}{(-1)-2} = \frac{-6}{-3} = \boxed{2}$$

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Example Continued

In order to find C_2, we will need to calculate the derivitive of $Q(s)$

$$= \frac{d}{dx}\left(\frac{6x^2+11x-1}{x-2}\right)$$

$$= \frac{12x^2+11}{x-2} + \frac{-1(6x^2+11x-1)}{(x-2)^2}$$

$$= \boxed{\frac{12x^2+11}{x-2} - \frac{6x^2+11x-1}{(x-2)^2}}$$

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Example Continued

$$= \frac{6x^2+11x-1}{(x-2)\,\cancel{(x+1)^2}} = \frac{5}{(x-2)} + \frac{C_2}{x+1} + \frac{2}{(x+1)^2}$$

$$C_2 = \frac{1}{(k-1)!}\left(\frac{d^{k-1}}{dx^{k-1}}\left(\frac{6x^2+11x-1}{x-2}\right)\right)\Bigg|_{x=-1} \qquad (k=2)$$

$$= \frac{1}{(2-1)!}\left(\frac{d^{2-1}}{dx^{2-1}}\left(\frac{6x^2+11x-1}{x-2}\right)\right)\Bigg|_{x=-1}$$

$$= (1)\left(\frac{d^1}{dx^1}\left(\frac{6x^2+11x-1}{x-2}\right)\right)\Bigg|_{x=-1}$$

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Example Continued

The derivative was found on the previous slide to be

$$\frac{d^1}{dx^1} = \frac{12x^2+11}{x-2} - \frac{6x^2+11x-1}{(x-2)^2}$$

$$C_2 = (1)\left(\frac{12x+11}{x-2} - \frac{6x^2+11x-1}{(x-2)^2}\right)\Bigg|_{x=-1}$$

$$= \frac{12(-1)+11}{-1-2} - \frac{6(-1)^2+11(-1)-1}{(-1-2)^2}$$

$$= \frac{-1}{-3} - \frac{-6}{(-3)^2} = \frac{1}{3} - \frac{-6}{9} = \frac{3}{9} + \frac{6}{9} = \boxed{1}$$

Now let's finish the integration

$$\int\left(\frac{5}{(x-2)} + \frac{1}{x+1} + \frac{2}{(x+1)^2}\right)dx$$

$$= \int\frac{5}{(x-2)}dx + \int\frac{1}{x+1}dx + \int\left(\frac{2}{(x+1)^2}\right)dx$$

$$= 5\int\frac{1}{(x-2)}dx + \int\frac{1}{x+1}dx + 2\int\left(\frac{1}{(x+1)^2}\right)dx$$

$$= \boxed{5\ln|x-2| + \ln|x+1| - \frac{2}{x+1} + C}$$

Area Example

$f_{top}(x)$: length = $3x - x$ width = dx $0<x<2$

$f_{bottom}(x)$: length = $6 - x$ width = dx $2<x<6$

$$\text{area} = \int_0^2 (3x-x)\,dx + \int_2^6 (6-x)\,dx$$

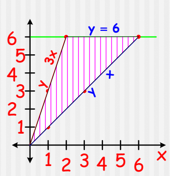
$$= \int_0^2 (2x)\,dx + \int_2^6 (6-x)\,dx$$

$$= x^2\Big|_0^2 + \left(6x - \frac{x^2}{2}\right)\Bigg|_2^6$$

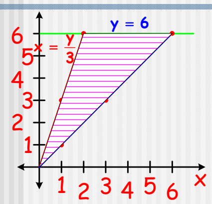
$$= (2^2 - 0) + \left[\left(6(6) - \frac{6^2}{2}\right) - \left(6(2) - \frac{2^2}{2}\right)\right]$$

$$= 4 + [(36-18) - (12-2)]$$

$$= 4 + [18-10] = \boxed{12}$$



Same Prob. integrated over y



Rewrite equation in terms of y

$$A = \int_0^6 \left(y - \frac{1}{3} y \right) dy$$

$$= \frac{2}{3} \int_0^6 y dy = \frac{2}{3} \left(\frac{y^2}{2} \right)_0^6$$

$$= \frac{2}{6} \left[6^2 - 0 \right] = \frac{2}{6} [36] = \boxed{12}$$

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Another Area example

Find the area between

$y_1 = \frac{1}{4} x + 3$ and $y_2 = 6x - 1$

between $x = 0$ and $x = \frac{1}{2}$

$$\text{Area} = \int_0^{1/2} \left[\left(\frac{1}{4} x + 3 \right) - (6x-1) \right] dx = \int_0^{1/2} \left[\left(\frac{1}{4} x + 3 \right) - (6x-1) \right] dx$$

$$= \int_0^{1/2} \left(-\frac{23}{4} x + 4 \right) dx = \left(-\frac{23}{4(2)} x^2 + 4x \right)_0^{1/2}$$

$$= \left(-\frac{23}{8} (1/2)^2 + 4(1/2) \right) - 0 = -\frac{23}{8(4)} + \frac{4[8(2)]}{2[8(2)]} = -\frac{23}{32} + \frac{64}{32} = \boxed{\frac{41}{32}}$$

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Average Value

The average value is the definite integral divided by the width of the boundary

$$\text{Average} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

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Average Value Example

What is the average value of

$y(x) = 2x + 4$ between $x = 0$ and $x = 4$?

$$\text{average} = \frac{1}{4-0}\int_0^4 (2x+4)\,dx$$

$$= \frac{1}{4}\left(\frac{2x^2}{2} + 4x\right)\Bigg|_0^4 = \frac{1}{4}\left(\frac{2(4)^2}{2} + 4(4)\right) - 0$$

$$= \frac{1}{4}(16+16) = \frac{32}{4} = \boxed{8}$$

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Work

w = force • distance

Example 1

Lift a 10 1b box 15ft. How much work was performed?

$w = (10\text{lb})(15') = \boxed{150\text{ft} - \text{lbs}}$

This was easy due to the fact that nothing changed during the problem.

Lets look at another example.

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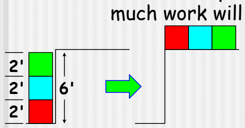
Work Example 2

There are three 20 lb boxes which have to be lifted to the top of a 6ft ledge. How much work will be performed?



Note that distance changes during the problem

box 1 ⇒ 2ft	w = 2'• 20 lbs	= 40ft – lbs
box 2 ⇒ 4ft	w = 4'• 20 lbs	= 80ft – lbs
box 3 ⇒ 6ft	w = 6'• 20 lbs	= 120ft – lbs
		240ft – lbs

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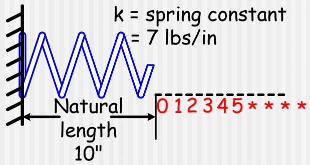
Work related to a spring

Hooks Law

$f = kx$

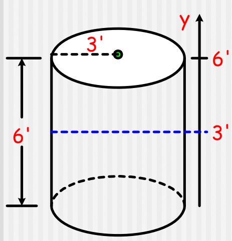


How much work is performed when the spring is stretched from 12" to 15"

$$w = \int_{12-10}^{15-10} 7x\,dx = \int_{2}^{5} 7x\,dx = \frac{7}{2}x^2\Big|_2^5 = \frac{7}{2}\left[5^2 - 2^2\right]$$

$$= \frac{7}{2}[25-4] = \frac{7}{2}[21] = \boxed{73.5 \text{ in} - \text{lbs}}$$

Another work example



Pump water out of the top of the tank till it is half full. How much work will be performed?

Method

Divide water into slabs

Force = weight of slab

Distance = how far slab is lifted

Example Continued

water density = $62.4\ \text{lbs}/\text{ft}^3$

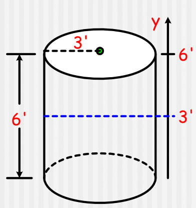
(from reference manual)

thickness = dy

radius = 3 ft

distance = 6 – y

limits of integration 3 ⇔ 6



Example Continued

$$\text{weight} = \left(62.4\,{}^{\text{lbs}}\!/_{\text{ft}^3}\right)3^2\pi\ dy$$

$$w = \int_3^6 \left(62.4\,{}^{\text{lbs}}\!/_{\text{ft}^3}\right)9\pi(6-y)\ dy$$

$$= 1,764.32\int_3^6 (6-y)\ dy$$

$$= 1,764.32\left(6y - \frac{y^2}{2}\right)\Bigg|_3^6$$

$$= 1,764.32\left[\left(6(6) - \frac{6^2}{2}\right) - \left(6(3) - \frac{3^2}{2}\right)\right]$$

$$= 1,764.32\left[(36-18) - (18-4.5)\right]$$

$$= 1,764.32[18 - 13.5] = 1,764.32[4.5] = \boxed{7,939.43\text{ft} - \text{lbs}}$$

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