Integrals

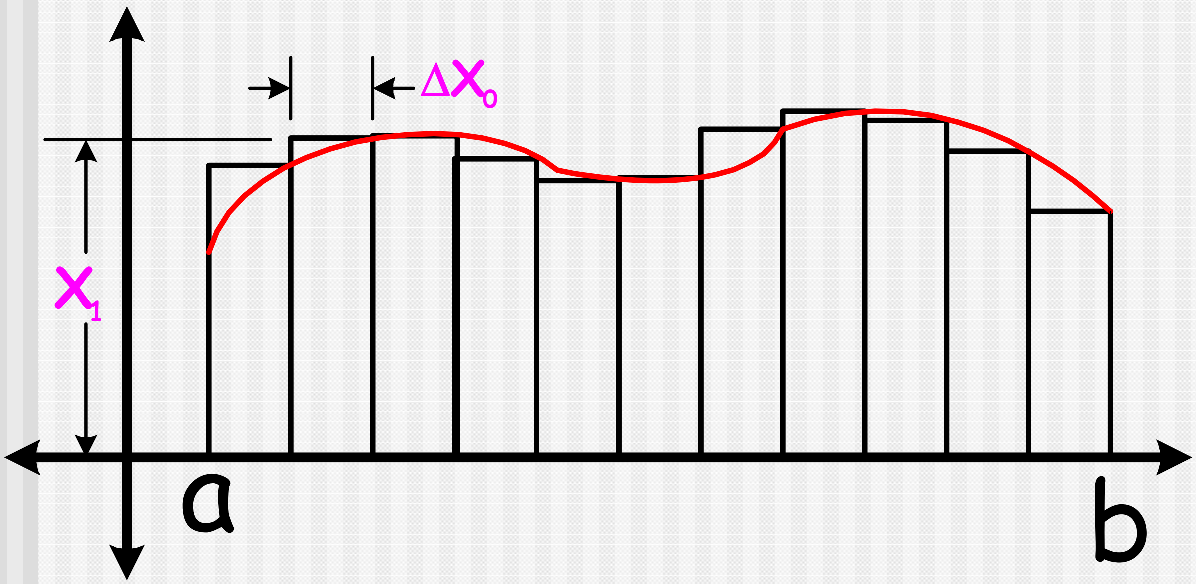
- Define Integrals
- Techniques of Integration
- Applications
  - Area
  - Work

Quote

You never really understand mathematics, you just get used to it.

John von Neumann
Famous mathematician

Integrals defined



a and b are the boundaries

$\Delta x \equiv$ width

$x_i \equiv$ height

$\|P\| \equiv$ Norm of the Partition = largest Δx

$$\text{Area} = \lim_{\|P\| \to 0} \left(\sum_{i=0}^{n} f(x)\,\Delta x_i \right) = \int_a^b f(x)\,dx$$

Fundamental Theorem of Calculus

$$D_x\left[F(x)\right] = f(x) \qquad F(x) \equiv \text{anti}-\text{derivative}$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int f(x)\,dx = F(x) + C \qquad \int \equiv \text{Indefinite integral}$$

$\Uparrow$

find the anti-derivative

Rules

$$\int u^r du = \frac{u^{r+1}}{r+1} + C \qquad \text{if} \quad r \neq -1$$

$$\int u^{-1} du = \int \frac{du}{u} = \ln|u| + C$$

$$\int \sin u \; du = -\cos u + C$$

$$\int \cos u \; du = \sin u + C$$

$$\int e^u du = e^u + C$$

Integration Methods

$$\int [af(u) + bg(u)]du = a\int f(u)\,du + b\int g(u)\,du$$

$$\int f(u)\,du$$

Must match exactly

u substitution

$$\int \left(f\left[g\left(x \right) \right] \bullet g'(x) \right) dx$$

$$\text{let } u = g(x) \qquad \therefore \frac{du}{dx} = g'(x)$$

$$\text{and} \qquad du = g'(x)dx$$

$$\Rightarrow \int f\left(u \right) du$$

u substitution example

$$\int \left(\sin x^3\right)\ x^2 dx$$

let $u = x^3$ so $du = 3x^2 dx$

This puts a 3 into the equation so we will have to multiply by $\frac{1}{3}$ to take it out

$$\frac{1}{3}\int \left(\sin u\right)\ du = \frac{-1}{3}\cos u + C = \boxed{\frac{-1}{3}\cos x^3 + C}$$

Another 'u' substitution Example

$$\int \frac{(\ln x)^3}{x} dx$$

$$\text{let } u = \ln x \quad \therefore \quad du = \frac{1}{x} dx$$

$$\int u^3 du = \frac{1}{4} u^4 + C = \boxed{\frac{1}{4}(\ln x)^4 + C}$$

Still another 'u' substitution example

Find $\int \left(e^{x} + 2x\right)^{2}\left(e^{x} + 2\right)dx$

let $u(x) = e^{x} + 2x$

so $du = \left(e^{x} + 2\right)dx$

$$= \int \left(e^{x} + 2x\right)^{2}\left(e^{x} + 2\right)dx$$

$$= \int u^{2}du = \frac{u^{3}}{3} + C$$

$$= \boxed{\frac{1}{3}\left(e^{x} + 2x\right)^{3} + C}$$

Integration by parts

$$\boxed{\int u\,dv = u \cdot v - \int v\,du}$$

needs u, v, and du

Example Solve $\int x \sin x\,dx$

$$\text{let} \begin{cases} u = x & dv = \sin x\,dx \\ du = dx & v = -\cos x \end{cases}$$

$$= -x\cos x - \int -\cos x\,dx = -x\cos x + \int \cos x\,dx$$

$$= \boxed{-x\cos x + \sin x + C}$$

Another "by parts" example

$$\int x^2 \ln x\,dx \Rightarrow \text{let} \quad \boxed{\begin{array}{ll} u = \ln x & dv = x^2 dx \\ du = \dfrac{1}{x} dx & v = \dfrac{1}{3} x^3 \end{array}}$$

$$uv - \int v\,du$$

$$\int x^2 \ln x\,dx = \frac{1}{3} x^3 \ln x - \int \left(\frac{1}{3} x^3\right)\left(\frac{1}{x} dx\right)$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 dx$$

$$= \boxed{\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C}$$

Still another "By parts" example

Find $\int x^2 e^x dx$

Let
$$u = x^2 \quad dv = e^x dx$$
$$du = 2x \quad v = e^x$$

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx$$
$$= x^2 e^x - 2\int x e^x dx$$

From handbook $\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1)$, so

$$\int x^2 e^x dx = x^2 e^x - 2e^x(x - 1) + C$$
$$= x^2 e^x - 2e^x\left(x e^x - e^x\right) + C$$

Partial Fraction Expansion

$$\int \frac{P(x)}{Q(x)}\,dx$$

$P(x)$ and $Q(x)$

are polynomials with an

order of $P(x) < Q(x)$

Partial Fraction Expansion Example

Solve $\int \frac{6x^2 + 11x - 1}{x^3 - 3x - 2} dx$

The denominator factors to $\Rightarrow (x+1)^2 (x-2)$

$$\frac{6x^2 + 11x - 1}{x^3 - 3x - 2} = \frac{A}{(x-2)} + \frac{C_2}{x+1} + \frac{C_1}{(x+1)^2}$$

Example Continued

$$\frac{6x^2+11x-1}{(x+1)^2(x-2)} = \frac{A}{(x-2)} + \frac{C_2}{x+1} + \frac{C_1}{(x+1)^2}$$

$$A = \left.\frac{6x^2+11x-1}{\cancel{(x-2)}(x+1)^2}\right|_{x=2} = \frac{6(2)^2+11(2)-1}{(2+1)^2}$$

$$= \frac{6(4)+22-1}{3^2}$$

$$= \frac{24+22-1}{3^2} = \frac{45}{9} = \boxed{5}$$

Partial Fraction Expansion Multiple Order Poles

$$F(x) = \frac{Q(x)}{(x+a)^i}$$

$$C_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}Q(x)\Big|_{x=-a}$$

Multiple Order Poles Example

$$\frac{6x^2+11x-1}{(x+1)^2(x-2)} = \frac{5}{(x-2)} + \frac{C_2}{x+1} + \frac{C_1}{(x+1)^2}$$

$$F(x) = \frac{Q(x)}{(x+a)^i} \Rightarrow C_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}Q(x)\bigg|_{x=-a}$$

Example Continued

$$C_1 = \frac{1}{(k-1)!}\left(\frac{d^{k-1}}{dx^{k-1}}\frac{6x^2+11x-1}{x-2}\right)\Bigg|_{x=-1} \qquad (k=1)$$

$$= \frac{1}{(1-1)!}\left(\frac{d^{1-1}}{dx^{1-1}}\frac{6x^2+11x-1}{x-2}\right)\Bigg|_{x=-1}$$

$$= (1)\left(\frac{d^0}{dx^0}\frac{6x^2+11x-1}{x-2}\right)\Bigg| = \left(\frac{6x^2+11x-1}{x-2}\right)\Bigg|_{x=-1}$$

$$= \frac{6(-1)^2+11(-1)-1}{(-1)-2} = \frac{-6}{-3} = \boxed{2}$$

Example Continued

In order to find C_2, we will need to calculate the derivitive of $Q(s)$

$$= \frac{d}{dx}\left(\frac{6x^2 + 11x - 1}{x - 2}\right)$$

$$= \frac{12x^2 + 11}{x - 2} + \frac{-1\left(6x^2 + 11x - 1\right)}{(x - 2)^2}$$

$$= \boxed{\frac{12x^2 + 11}{x - 2} - \frac{6x^2 + 11x - 1}{(x - 2)^2}}$$

Example Continued

$$= \frac{6x^2 + 11x - 1}{(x-2)\cancel{(x+1)^2}} = \frac{5}{(x-2)} + \frac{C_2}{x+1} + \frac{2}{(x+1)^2}$$

$$C_2 = \frac{1}{(k-1)!}\left(\frac{d^{k-1}}{dx^{k-1}}\left(\frac{6x^2 + 11x - 1}{x-2}\right)\right)\Bigg|_{x=-1} \qquad (k=2)$$

$$= \frac{1}{(2-1)!}\left(\frac{d^{2-1}}{dx^{2-1}}\left(\frac{6x^2 + 11x - 1}{x-2}\right)\right)\Bigg|_{x=-1}$$

$$= (1)\left(\frac{d^{1}}{dx^{1}}\left(\frac{6x^2 + 11x - 1}{x-2}\right)\right)\Bigg|_{x=-1}$$

Example Continued

The derivative was found on the previous slide to be

$$\frac{d^1}{dx^1} = \frac{12x^2 + 11}{x-2} - \frac{6x^2 + 11x - 1}{(x-2)^2}$$

$$C_2 = (1)\left(\frac{12x + 11}{x-2} - \frac{6x^2 + 11x - 1}{(x-2)^2}\right)\Bigg|_{x=-1}$$

$$= \frac{12(-1) + 11}{-1 - 2} - \frac{6(-1)^2 + 11(-1) - 1}{(-1-2)^2}$$

$$= \frac{-1}{-3} - \frac{-6}{(-3)^2} = \frac{1}{3} - \frac{-6}{9} = \frac{3}{9} + \frac{6}{9} = \boxed{1}$$

Now let's finish the integration

$$\int\left(\frac{5}{(x-2)}+\frac{1}{x+1}+\frac{2}{(x+1)^2}\right)dx$$

$$=\int\frac{5}{(x-2)}dx+\int\frac{1}{x+1}dx+\int\left(\frac{2}{(x+1)^2}\right)dx$$

$$=5\int\frac{1}{(x-2)}dx+\int\frac{1}{x+1}dx+2\int\left(\frac{1}{(x+1)^2}\right)dx$$

$$=\boxed{5\ln|x-2|+\ln|x+1|-\frac{2}{x+1}+C}$$

Area Example

$f_{top}(x)$: length $= 3x - x$ width $= dx$ $0<x<2$

$f_{bottom}(x)$: length $= 6 - x$ width $= dx$ $2<x<6$

$$\text{area} = \int_0^2 (3x - x)\,dx + \int_2^6 (6 - x)\,dx$$

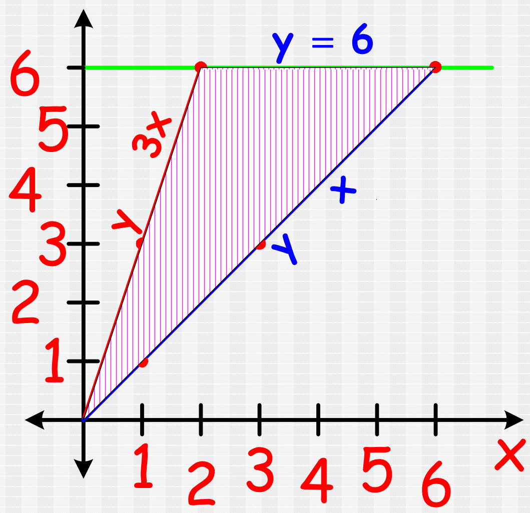
$$= \int_0^2 (2x)\,dx + \int_2^6 (6 - x)\,dx$$

$$= x^2\Big|_0^2 + \left(6x - \frac{x^2}{2}\right)\Bigg|_2^6$$

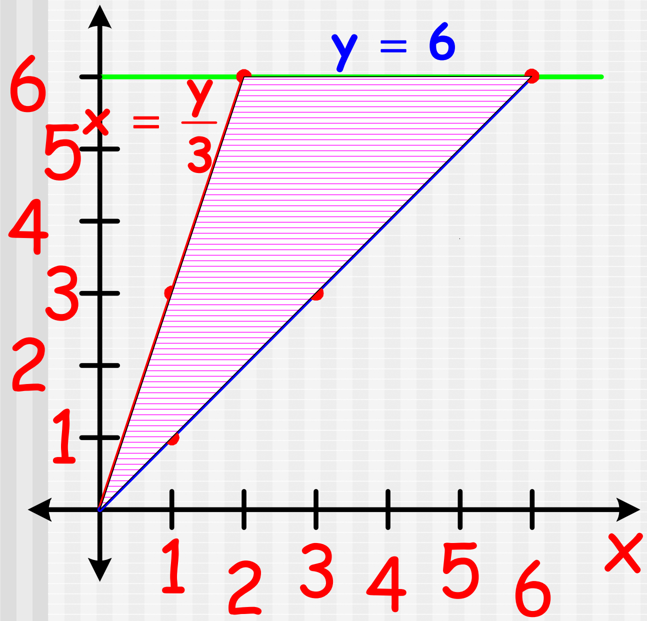
$$= (2^2 - 0) + \left[\left(6(6) - \frac{6^2}{2}\right) - \left(6(2) - \frac{2^2}{2}\right)\right]$$

$$= 4 + [(36 - 18) - (12 - 2)]$$

$$= 4 + [18 - 10] = \boxed{12}$$



Same Prob. integrated over y



Rewrite equation in terms of y

$$A = \int_0^6 \left(y - \frac{1}{3} y \right) dy$$

$$= \frac{2}{3} \int_0^6 y dy = \frac{2}{3} \left(\frac{y^2}{2} \right)_0^6$$

$$= \frac{2}{6} \left[6^2 - 0 \right] = \frac{2}{6} [36] = \boxed{12}$$

Another Area example

Find the area between

$$y_1 = \frac{1}{4}x + 3 \quad \text{and} \quad y_2 = 6x - 1$$

$$\text{between } x = 0 \text{ and } x = \frac{1}{2}$$

$$\text{Area} = \int_0^{1/2}\left[\left(\frac{1}{4}x + 3\right) - (6x-1)\right]dx = \int_0^{1/2}\left[\left(\frac{1}{4}x + 3\right) - (6x-1)\right]dx$$

$$= \int_0^{1/2}\left(-\frac{23}{4}x + 4\right)dx = \left(-\frac{23}{4(2)}x^2 + 4x\right)_0^{1/2}$$

$$= \left(-\frac{23}{8}(1/2)^2 + 4(1/2)\right) - 0 = -\frac{23}{8(4)} + \frac{4[8(2)]}{2[8(2)]} = -\frac{23}{32} + \frac{64}{32} = \boxed{\frac{41}{32}}$$

Average Value

The average value is the definite integral divided by the width of the boundary

$$\text{Average} = \frac{1}{b - a}\int_a^b f(x)\,dx$$

Average Value Example

What is the average value of

$y(x) = 2x + 4$ **between** $x = 0$ **and** $x = 4$ **?**

$$\text{average} = \frac{1}{4-0}\int_0^4 (2x+4)\,dx$$

$$= \frac{1}{4}\left(\frac{2x^2}{2} + 4x\right)\Bigg|_0^4 = \frac{1}{4}\left(\frac{2(4)^2}{2} + 4(4)\right) - 0$$

$$= \frac{1}{4}(16+16) = \frac{32}{4} = \boxed{8}$$

Work

$w = \text{force} \bullet \text{distance}$

Example 1

Lift a 10 1b box 15ft. How much work was performed?

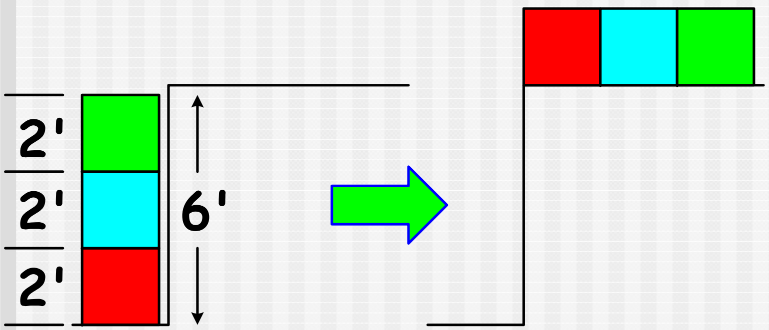
$w = (10\text{lb})(15') = \boxed{150\text{ft} - \text{lbs}}$

This was easy due to the fact that nothing changed during the problem.

Lets look at another example.

Work Example 2

There are three 20 lb boxes which have to be lifted to the top of a 6ft ledge. How much work will be performed?



Note that distance changes during the problem

$$\text{box } 1 \Rightarrow 2\text{ft} \quad w = 2' \cdot 20 \text{ lbs} = 40\text{ft} - \text{lbs}$$

$$\text{box } 2 \Rightarrow 4\text{ft} \quad w = 4' \cdot 20 \text{ lbs} = 80\text{ft} - \text{lbs}$$

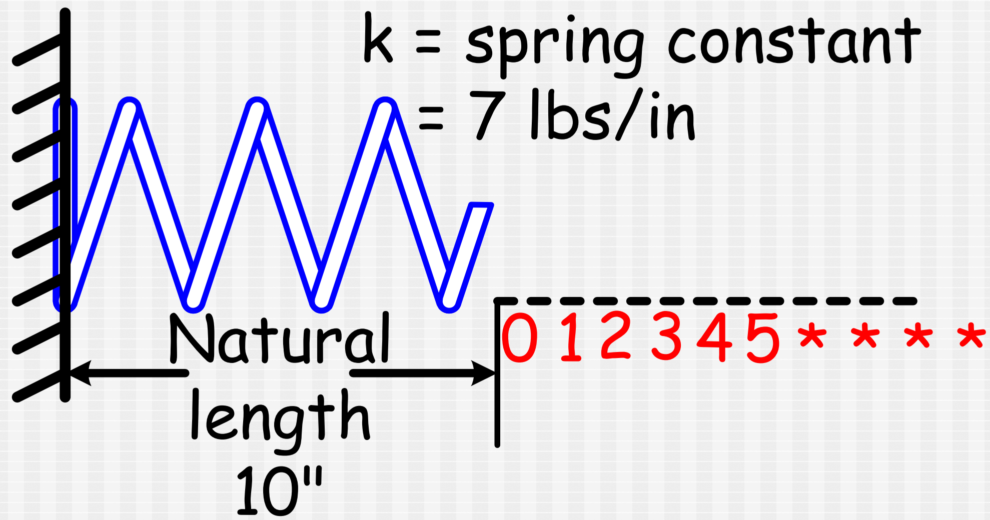
$$\text{box } 3 \Rightarrow 6\text{ft} \quad w = 6' \cdot 20 \text{ lbs} = 120\text{ft} - \text{lbs}$$

$$\boxed{240\text{ft} - \text{lbs}}$$

Work related to a spring

Hooks Law

$$f = kx$$

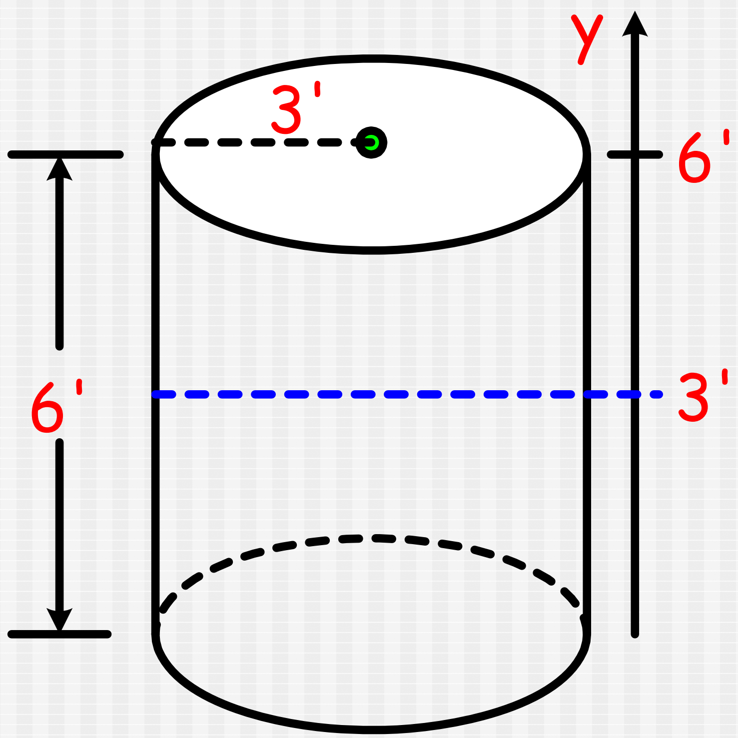


How much work is performed when the spring is stretched from 12" to 15"

$$w = \int_{12-10}^{15-10} 7x dx = \int_{2}^{5} 7x\ dx = \frac{7}{2}x^2\Big|_2^5 = \frac{7}{2}\left[5^2 - 2^2\right]$$

$$= \frac{7}{2}[25 - 4] = \frac{7}{2}[21] = \boxed{73.5\ in - lbs}$$

Another work example



Pump water out of the top of the tank till it is half full. How much work will be performed?

Method

Divide water into slabs

Force = weight of slab

Distance = how far slab is lifted

Example Continued

water density = $62.4\,{}^{lbs}\!/\!_{ft^3}$

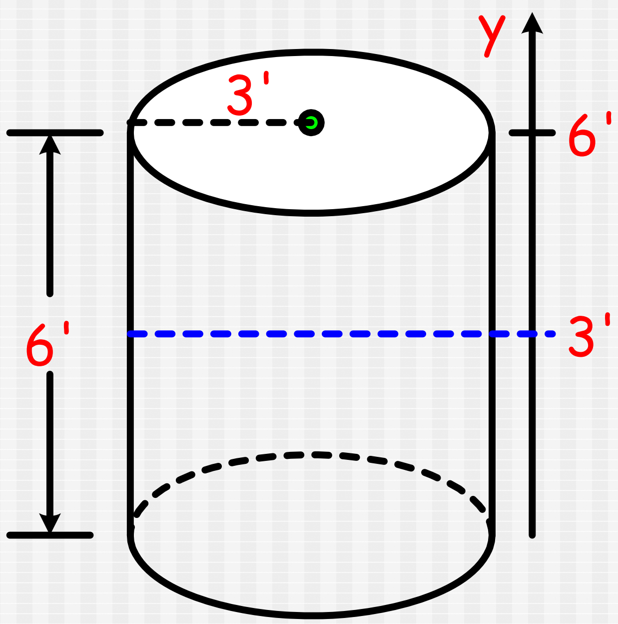
(from reference manual)

thickness = dy

radius = 3 ft

distance = 6 − y

limits of integration 3 ⇔ 6



Example Continued

$$\text{weight} = \left(62.4 \, {}^{\text{lbs}}\!/\!_{\text{ft}^3}\right) 3^2 \pi \; dy$$

$$w = \int_3^6 \left(62.4 \, {}^{\text{lbs}}\!/\!_{\text{ft}^3}\right) 9\pi (6 - y) \; dy$$

$$= 1,764.32 \int_3^6 (6 - y) \; dy$$

$$= 1,764.32 \left(6y - \frac{y^2}{2}\right)\Bigg|_3^6$$

$$= 1,764.32 \left[\left(6(6) - \frac{6^2}{2}\right) - \left(6(3) - \frac{3^2}{2}\right)\right]$$

$$= 1,764.32 \left[(36 - 18) - (18 - 4.5)\right]$$

$$= 1,764.32 [18 - 13.5] = 1,764.32 [4.5] = \boxed{7,939.43 \text{ft} - \text{lbs}}$$