Advanced Calculus

- Series and Infinite Series
- Vectors and Vector Value functions
- Multivariable Calculus
- Polar Coordinates

Infinite Series

$$\sum_{n=1}^{\infty} a_n$$

If the series adds up to a finite value, the series will converge. Convergence is the main interest in a series. Even if the series adds up to a finite value, the series still might not converge because the terms don't go to zero fast enough.

$$\lim_{n=\infty} a_n = 0$$

Arithmetic Progression

Subtract each number from the preceding (2nd – 1st) etc. If the difference is a constant, the series is arithmetic.

Or

Subtract the possible answer from the last number in the sequence. If the the difference is the same, then that answer is the correct answer.

Arithmetic Progression Example

What is the next number in the sequence

[14,17,20,23,...]

14 + 3 = 17 + 3 = 20 + 3 = 23 =,...

The series has a difference of +3 between each number, so the next number will be +26.

Geometric Progression

Divide each number by the

preceeding number, i.e. $2^{nd}/1^{st}$.

If the qoutients are equal,

the series is geometric.

Or

If any of the possible answers

are integer multiples of the last

number, try that number on

the others in the series.

Geometric Progression Example

What is the next number is the sequence $[3, 21, 147, 1029, ...]$

$3 \bullet 7 = 21 \bullet 7 = 147 \bullet 7 = 1029 \bullet 7 = \mathbf{7203}$

Arithmetic Series

$$L = n^{th} \text{ term} = a + (n - 1)d$$

$$S = [\text{summation terms } L \text{ through } n]$$

$$= \frac{n(a + L)}{2} = \frac{n[2a + (n - 1)d]}{2}$$

What is the summation of the series

$$3 + \frac{(n - 1)}{7} \quad \text{for } 4 \text{ terms?}$$

$$S = \frac{n[2a + (n - 1)d]}{2} = \frac{4[2(3) + (4 - 1)7]}{2}$$

$$= 2[6 + (3)7] = 2[6 + 21] = 2[27] = \boxed{54}$$

Geometric Series

$a_n = ar^n$ series will converge if $|r| < 1$

Example :

$3 + \frac{3}{2} + \frac{3}{4} + \ldots$ Each value is $\frac{1}{2} \bullet$ previous value

Power Series

$$a_n = \frac{1}{n^p}$$ series will converge if $p > 1$

Example :

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Power Series

$$a_n = k_n (x - x_0)^n$$

Test for convergence $\Rightarrow \lim_{n \Rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

Example :

$$a_n = \frac{n}{5^n} x^n$$

$$= \lim_{n \Rightarrow \infty} \left| \frac{(n+1)\, x^{n+1}}{5^{n+1}} \bullet \frac{5^n}{n x^n} \right|$$

$$= \lim_{n \Rightarrow \infty} \left(\frac{n+1}{n} \right) \left| \frac{x}{5} \right| < 1$$

$$-5 < x < 5$$

Taylor series

$$K_n = \frac{f^{(n)}(x_0)}{n!}$$

Vectors

Scalars have magnitude only
Vectors have both magnitude and direction

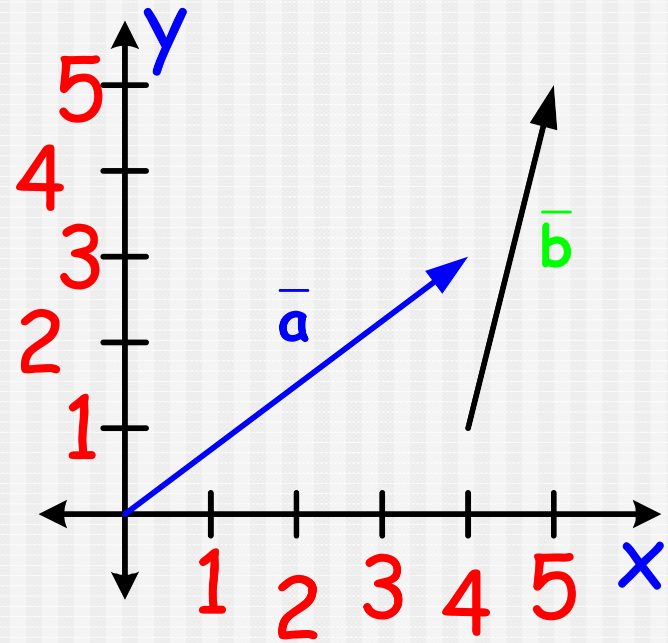
$$\bar{a} = \langle 4, 3 \rangle$$

$$|\bar{a}| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9}$$

$$= \sqrt{25} = \boxed{5}$$

$$\bar{b} = \langle 1, 4 \rangle$$

$$|\bar{b}| = \sqrt{1^2 + 4^2} = \sqrt{1 + 16} = \boxed{\sqrt{17}}$$



Vector operations

add vectors $\Rightarrow \bar{a} + \bar{b} = \langle 4, 3 \rangle + \langle 1, 4 \rangle = \boxed{\langle 5, 7 \rangle}$

mult by scalarvectors $\Rightarrow 2\bar{b} = 2\langle 1, 4 \rangle = \boxed{\langle 2, 8 \rangle}$

Unit Vector

$$\frac{\bar{a}}{|\bar{a}|} = \frac{\langle 4, 3 \rangle}{5} = \boxed{\left\langle \frac{4}{5}, \frac{3}{5} \right\rangle} = \hat{a}$$

$$\boxed{\hat{i} = \langle 1, 0, 0 \rangle \quad \hat{j} = \langle 0, 1, 0 \rangle \quad \hat{k} = \langle 0, 0, 1 \rangle}$$

$$\bar{b} = \langle 1, 4 \rangle = \hat{i} + 4\hat{j}$$

Dot Product

$$\bar{a} = \langle a_1, a_2, a_3 \rangle \quad \bar{b} = \langle b_1, b_2, b_3 \rangle$$

$$\bar{a} \bullet \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= |\bar{a}|\,|\bar{b}| \cos\theta$$

θ is angle between the two vectors

Ex:

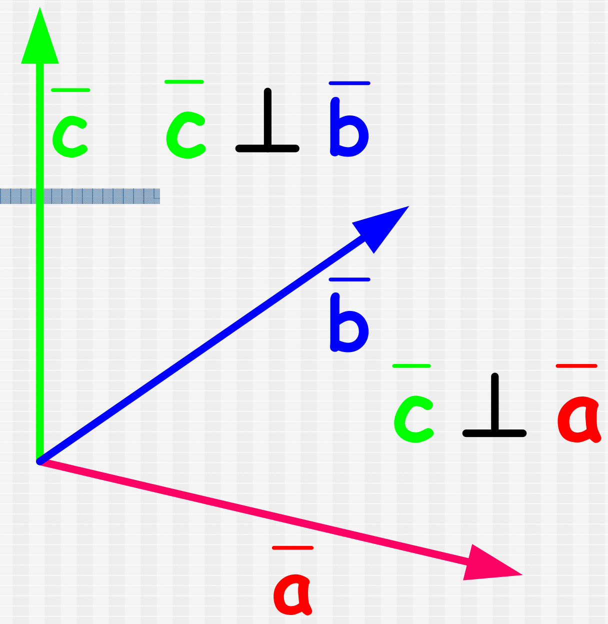
$$|\bar{a}| = \langle 4, 3 \rangle \quad \bar{b} = \langle 1, 4 \rangle$$

$$\bar{a} \bullet \bar{b} = 4 \bullet 1 + 3 \bullet 4 = 4 + 12 = 16$$

$$\cos\theta = \frac{\bar{a} \bullet \bar{b}}{|\bar{a}|\,|\bar{b}|} = \frac{16}{5\sqrt{17}} = \boxed{0.776}$$

$$\theta = \boxed{39.1^\circ}$$

Cross Product



$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\hat{i}(a_2 b_3 - a_3 b_2) - \hat{j}(a_1 b_3 - a_3 b_1) + \hat{k}(a_1 b_2 - a_2 b_1) = \bar{c}$$

$$\bar{c} = |\bar{a}|\,|\bar{b}| \sin\theta$$

Right hand rule:

Curl your fingers in the direction of $\bar{A}$ to $\bar{B}$.

The thumb will point in the direction of $\bar{C}$.

Vector Function

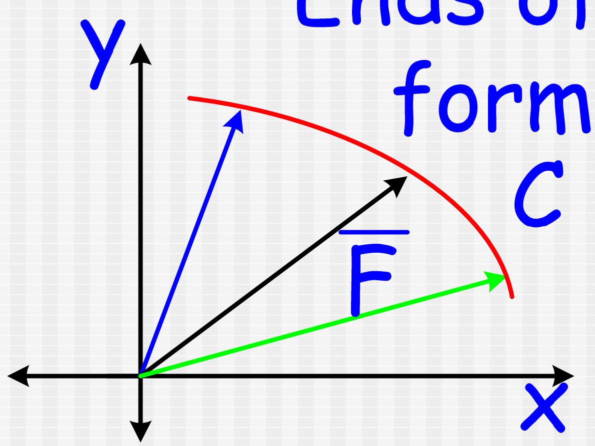
$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$\bar{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$$

same type of rule applies to integration of Vectors

Forming a trace thru space

Ends of the vectors
form the curve



$\bar{r}(t)$ = position(time)

$\bar{r}\,'(t)$ = velocity(time)

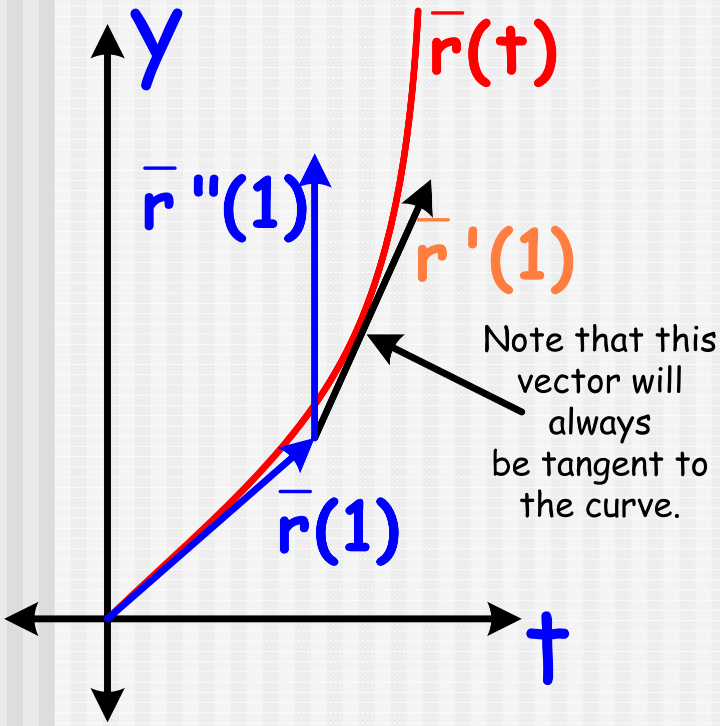
$\bar{r}\,''(t)$ = acceleration(time)

Vector example

$y = x^2$

$X(t) = t \Rightarrow y(t) = t^2$

(called 'parameterizing the curve')



$$\bar{r}(t) = t\hat{i} + t^2\hat{j}$$

$$\bar{r}(1) = \hat{i} + \hat{j}$$

$$\bar{r}'(t) = \hat{i} + 2t\hat{j}$$

$$\bar{r}'(1) = \hat{i} + 2\hat{j}$$

$$\bar{r}''(t) = 2\hat{j}$$

$$\bar{r}''(1) = 2\hat{j}$$

One function of several variables

$$z = f(x, y) \qquad T = f(x, y, z)$$

Example : $f(x, y) = x^2 + y + e^{x+y}$

Derivatives are taken with respect to (w.r.t.) something. The question here is: w.r.t what?

"Partial Derivative"

$$f_x = \frac{\partial f(x, y)}{\partial x}$$ means w.r.t x

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f(x, y)}{\partial x}\right)$$ means 1st w.r.t x then w.r.t. y

Partial Derivative example

$$f(x, y) = x^2 + y + e^{xy}$$

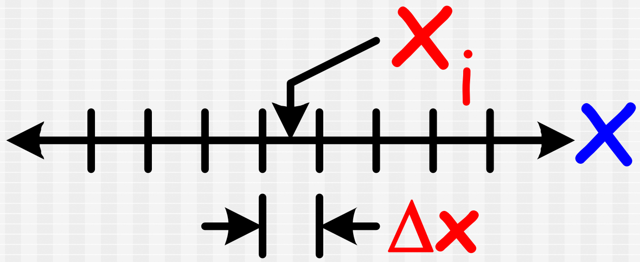
$$f_x = 2x + ye^{xy}$$

$$f_y = 1 + xe^{xy}$$

$$f_{xy} = e^{xy} + y\left(xe^{xy}\right)$$

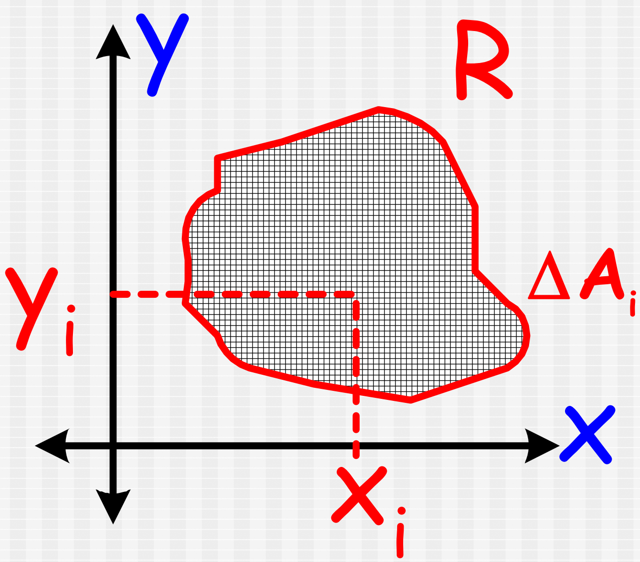
Multi-Variable Integration

Remember when we integrated with One variable we selected an interval As shown to the right?



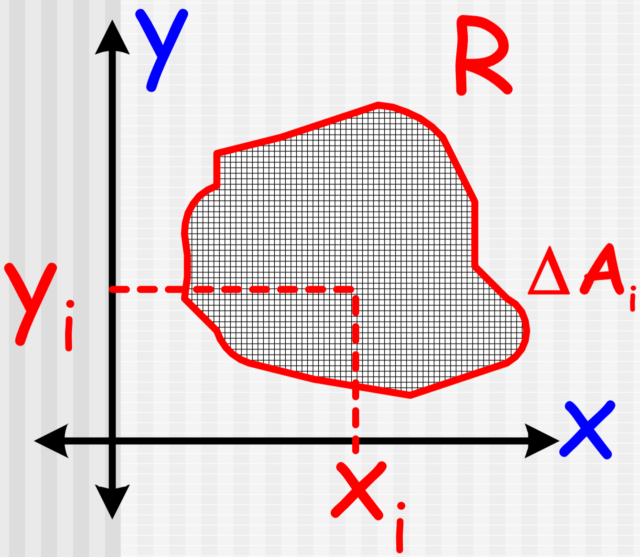
$$\sum f(x_i)\Delta x_i$$

$$\Rightarrow \int_a^b f(x)\,dx$$



When add on a second variable to integrate over we are now talking about a region. It is no longer possible to plot the function on paper.

Integration (cont)



$$\sum f(x_i, y_i)\Delta A_i$$

$$\iint_R f(x, y)\, dA \quad \text{where } dA = dxdy$$

$$\int \left[\int f(x, y)\, dx \right] dy$$

curve boundaries over the region are the boundaries of integration.

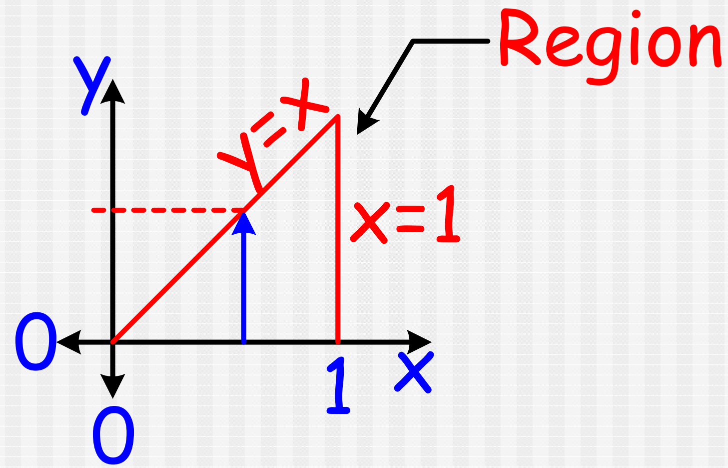
Example

$$f(x, y) \equiv x^2 + y$$

$$= \int_0^1 \int_0^x (x^2 + y)\,dy\,dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^2}{2} \right) \Big|_0^x dx$$

$$= \int_0^1 \left[\left(x^2 (x) + \frac{(x)^2}{2} \right) - \left(x^2 (0) + \frac{(0)^2}{2} \right) \right] dx$$



The arrow points to the inside variable

Example continued

$$= \int_0^1 \left[\left(x^2(x) + \frac{(x)^2}{2} \right) - \left(x^2(0) + \frac{(0)^2}{2} \right) \right] dx$$

$$= \int_0^1 \left(x^3 + \frac{x^2}{2} \right) dx$$

$$= \frac{x^4}{4} + \frac{x^3}{2(3)}$$

$$= \left(\frac{x^4}{4} + \frac{x^3}{6} \right) \Bigg|_0^1$$

$$= \left(\frac{1^4}{4} + \frac{1^3}{6} \right) - \left(\frac{0^4}{4} + \frac{0^3}{6} \right) = \frac{1}{4} + \frac{1}{6} = \boxed{\frac{5}{12}}$$

Polar coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$x : (x, y) = \left(\sqrt{3}, 1\right) \Rightarrow \left(2, \frac{\pi}{6}\right)$$

$$dA = (rd\theta)\,dr$$

$$= rdrd\theta$$