Differential Equations

A **Differential Equation (DE)** is an equation containing one or more derivatives of an unknown dependant variable with respect to (wrt) one or more independent variables. Solution of the **DE** involves determining the unknown variable. The solution of a **DE** involves the use of boundary conditions.

Differential equations arise in virtually all areas of engineering, physics, and most other sciences. Many of these equations are to complex for a closed-form solution and require **numerical approximation techniques.**

Definition: Equations involving one or more derivatives of y where $y = f(t)$

Order: Highest derivative which appears in the equation.

Linear Differential Equations: A D.E. is linear when $y(t)$ and its derivatives are multiplied by only constants or by known functions of t. (math classes usually use x instead of t.)

Some Differential Equation classifications

Linear	
Constant Coefficient	Time-varying coefficients
Ordinary	
Function of one variable (such as t)	

Classifications continued

Non-linear
Coefficients are functions of y and/or higher powers of derivatives
Partial
Function of 2 or more variables (such as t, x, etc) $\frac{\partial^2 y}{dt^2}$, $\frac{\partial^2 y}{dx^2}$, $\frac{\partial^2 y}{dxdt}$, etc

The preceding definitions relate to "**continuous-time**" or "**analog**" systems. However, the same forms may be adapted to "**discrete-time**" or "**digital**" systems. In fact, the computer solutions of **DE**'s involves representing the analog functions by **digital approximations**.

Which of the following equations is linear?

$$y''(t) + 5ty'(t) - te^{-t}y(t) = 2t$$

or

$$y'''(t) + 2y(t)\,y'(t) - \left(y(t)\right)^2 = e^{-t}$$

The first equation is linear. Note that the derivatives of y are multiplied by known values of t. The second equation is a third order, non-linear circuit. It's derivatives of y are multiplied by unknown functions of t.

Most theory of Differential equations assumes the function to be linear.

Linear Constant Coefficient Ordinary Differential Equations (LCCODE)

The most common type of DE encountered in routine engineering applications is the LCCODE. It's General Form is:

$$f(t) = b_m \frac{d^m y}{dt^m} + b_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + \ldots + b_1 \frac{d^1 y}{dt} + b_0 y$$

The integer 'm' is the order of the DE

$$f(t) = b_m \frac{d^m y}{dt^m} + b_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + \ldots + b_1 \frac{d^1 y}{dt} + b_0 y$$

When $f(t) = 0$, the DE is said to be homogeneous.

When $f(t) \neq 0$, the DE is said to be non-homogeneous.

To completely solve the DE of "order 'm', it is necessary to specify 'm' boundary conditions. In many cases, these are:

$$y(0), y'(0), y''(0), \ldots, y^{m-1}(0)$$

LCCODE Solution

The general solution y of an LCCODE is of the form:

$$y = y_c + y_p$$

where

$$y_c \doteq \text{the complementary solution}$$

$$y_p \doteq \text{the particular solution}$$

The complementary solution is obtained from it's homogeneous equation by momentarily setting $f(t) = 0$.

It's form exhibits the natural properties of the system

$$y = y_c + y_p$$

where

$$y_c \doteq \text{the complementary solution}$$

$$y_p \doteq \text{the particular solution}$$

The particular solution depends on the form of $f(t)$ and is determined after the preceding step.

Boundary conditions are applied to the combination y as shown above.

Homogeneous Equations

Since the complementary solution is obtained from the homogeneous equation, we begin with:

$$f(t) = b_m \frac{d^m y}{dt^m} + b_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + \ldots + b_1 \frac{d^1 y}{dt} + b_0 y$$

all solutions are of the form Ce^{pt}

$$y = Ce^{pt}$$

$$\frac{dy}{dt} = pCe^{pt}$$

$$\vdots$$

$$\frac{d^m y}{dt^m} = p^m Ce^{pt}$$

Substitution of the various derivatives into the general equation yields:

$$b_m p^m Ce^{pt} + b_{m-1} p^{m-1} Ce^{pt} + \ldots + b_1 p Ce^{pt} + b_0 = 0$$

dividing thru by Ce^{pt}

$$b_m p^m + b_{m-1} p^{m-1} + \ldots + b_1 p + b_0 = 0$$

$$b_m p^m + b_{m-1} p^{m-1} + \ldots + b_1 p + b_0 = 0$$

This is called the "Characteristic Equation"

Let $p_1, p_2, \ldots p_m$ represent the 'm' roots.

Then

$C_1 e^{p_1 t}$ is a solution

$C_2 e^{p_2 t}$ is a solution

etc

The complementary solution is now a linear combination of the form:

$$y_c = C_1 e^{p_1 t} + C_2 e^{p_2 t} + \ldots + C_m e^{p_m t}$$

The particular solution, y_p, has the form of it's excitation. This solution is determined INDEPENDENT of the complementary solution. Some forms are:

Form of f(t)	Form assumed for y_p
Constant	A
xt	$A_1 t + A_0$
$\sin \omega t$ or $\cos \omega t$	$A \sin \omega t$ or $B \cos \omega t$

Example 1 (y_c)

$y' + 2y = 0 \qquad y(0) = 10$

$y_c = y = Ce^{pt}$

$y_c' = Cpe^{pt}$

substituting into the DE,

$Cpe^{pt} + 2Ce^{pt} = 0$

dividing both sides by Ce^{pt}

$p + 2 = 0 \therefore \boxed{p = -2}$

Example 1 (Solution)

Since the DE is equated to 0 there isn't a particular solution (y_p). All that is necessary now is to apply the boundary conditions to the expression.

$$y = y_c = Ce^{-2t}$$

$y(0) = 10$ so at $t = 0$,

$$y(0) = 10 = Ce^{-2(0)}$$

$$10 = C(1) \therefore \boxed{C = 10}$$

$$\boxed{y = 10e^{-2t}}$$

Example 2 (Y_c)

This expression non-homogeneous therefore it will have a complementary solution and a particular solution.

$$y' + 2y = 12 \qquad y(0) = 10$$

$$y_c = Ce^{pt}$$

$$y_c' = Cpe^{pt}$$

Substituting into the DE which has been set to 0,

$$Cpe^{pt} + 2Ce^{pt} = 0$$

Dividing both sides by Ce^{pt}

$$p + 2 = 0 \therefore \boxed{p = -2}$$

$$y_c = Ce^{-2t}$$

Example 2 (y_p)

$y' + 2y = 12 \quad y(0) = 10 \quad y_c = Ce^{-2t}$

Since the DE was non-homogeneous, it is necessary to determine a particular solution which is based on the form of the expression to which the DE is equated. In this case the expression is equated to a constant, so assume $y_p = A$, and $y_p' = 0$.

Substituting into the DE,

$12 = 0 + 2A \qquad \therefore A = 6$ so $\boxed{y_p = 6}$

The general solution is $\boxed{y = y_c + y_p = Ce^{-2t} + 6}$

Finally, applying the boundary conditions

$y(0) = 10$ so at $t = 0$,

$y(0) = 10 = Ce^{-2(0)} + 6$

$10 = C(1) + 6 \therefore \boxed{C = 4}$

$\boxed{y = 4e^{-2t} + 6}$

Example 3

$y' + 2y = 12\sin 3t \qquad y(0) = 10$

$y_c = y = Ce^{pt}$

$y' = Cpe^{pt}$

substituting,

$Cpe^{pt} + 2Ce^{pt} = 0$

dividing both sides by Ce^{pt}

$p + 2 = 0 \therefore \boxed{p = -2}$

$y_p = A\sin 3t + B\cos 3t$

$y_p' = 3A\cos 3t - 3B\sin 3t$

Example 3 (cont)

substituting

$$y_p = A\sin 3t + B\cos 3t$$

$$y_p' = 3A\cos 3t - 3B\sin 3t$$

into $12\sin 3t = y' + 2y$ we get

$$12\sin 3t = 3A\cos 3t - 3B\sin 3t + 2\left(A\sin 3t + B\cos 3t\right)$$

$$12\sin 3t = 3A\cos 3t - 3B\sin 3t + 2A\sin 3t + 2B\cos 3t$$

Example 3 (cont)

$$12\sin 3t = 3A\cos 3t - 3B\sin 3t + 2A\sin 3t + 2B\cos 3t$$

Equate equivalent items

$$0 = 3A\cos 3t + 2B\cos 3t$$

$$12\sin 3t = 2A\sin 3t - 3B\sin 3t$$

$$\begin{array}{l} (0 = 3A + 2B)\bullet 3 \\ (12 = 2A - 3B)\bullet 2 \end{array} \Rightarrow \begin{array}{l} 0 = 9A + 6B \\ 24 = 4A - 6B \\ \hline 24 = 13A \end{array}$$

$$A = \frac{24}{13} = \boxed{1.846}$$

$$0 = 9(1.846) + 6B$$

$$B = \boxed{-2.769}$$

Example 3 (cont)

$A = \boxed{1.846}$ $B = \boxed{-2.769}$

The General Solution is now:

$$y = Ce^{-2t} + 1.846 \sin 3t - 2.769 \cos 3t$$

Example 3 (cont)

$$y = Ce^{-2t} + 1.846\sin 3t - 2.769\cos 3t \qquad \text{(general solution)}$$

Now it is time to apply the boundary conditions:

$$y(0) = 10 = Ce^{-2t} + 1.846\sin 3t - 2.769\cos 3t$$

$$y(0) = 10 = Ce^{-2(0)} + 1.846\sin 3(0) - 2.769\cos 3(0)$$

$$10 = C(1) + 1.846(0) - 2.769(1)$$

$$10 = C - 2.769$$

$$C = 12.769$$

Thus,

$$\boxed{y = 12.769e^{-2t} + 1.846\sin 3t - 2.769\cos 3t}$$

Example 3 Solution Examined

The final solution was found to be:

$$y = 12.769e^{-2t} + 1.846\sin 3t - 2.769\cos 3t$$

However :

In the field of electronics, having a function with both a SIN as well as a COS in it is not very helpful. The signal you see on an o-scope is a combination of the two signals at some angle. Therefore, it is nice to combine them into a different (more helpful equation). (besides, the choice they give might not be the one above!)

Example 3 Solution Examined (cont)

$$y = 12.769e^{-2t} + 1.846\sin 3t - 2.769\cos 3t$$

$$Mag = \sqrt{(1.846)^2 + (-2.769)^2} = 3.328$$

$$angle = \tan^{-1}\left(\frac{-2.769}{1.846}\right) \quad \text{Note: 4th quadrant}$$

$$= -56.31°$$

$$\boxed{y = 12.769e^{-2t} + 3.328\sin\left(3t - 56.31°\right)}$$

Example 4

Solve :

$$y''(t) + 3y' + 2y = 0,$$

Boundary conditions: $y(0) = 10$, $y'(0) = 0$

The DE is homogeneous so there will only be a complementary solution, y_c.

$$p^2 + 3p + 2 = 0$$

Solving for the roots gives, we get:

$$P_1 = -1 \text{ and } P_2 = -2$$

Example 4

$$y_c = y = C_1 e^{-t} + C_2 e^{-2t}$$

$$y(0) = 10 = C_1 e^{-(0)} + C_2 e^{-2(0)}$$

$$10 = C_1(1) + C_2(1)$$

$$\boxed{10 = C_1 + C_2}$$

$$y_c' = -C_1 e^{-t} - 2C_2 e^{-2t}$$

$$y'(0) = 0 = -C_1 e^{-(0)} - 2C_2 e^{-2(0)}$$

$$\boxed{0 = -C_1 - 2C_2}$$

$$\begin{array}{rcrcr} 10 & = & C_1 & + & C_2 \\ 0 & = & -C_1 & + & -2C_2 \\ \hline 10 & = & & & -C_2 \end{array}$$

$$\boxed{C_2 = -10}$$ and $$\boxed{\begin{array}{l} 10 = C_1 + (-10) \\ \boxed{C_1 = 20} \end{array}}$$

$$\boxed{y(t) = 20e^{-t} - 10e^{-2t}}$$

Example 5

Solve :

$$y''(t) + 3y' + 2y = 24,$$

Boundary conditions: $y(0) = 10$, $y'(0) = 0$

The DE is now non-homogeneous therefore it will have both complementary and particular solutions.

Example 5 (y_c)

$$y'' + 3y' + 2y = 24, \quad y(0) = 10 \quad y'(0) = 0$$

For the Complementary solution, set the DE to 0 and find its roots.

$$p^2 + 3p + 2 = 0$$

The roots are $\Rightarrow P_1 = -1$ and $P_2 = -2$

The complementary solution form will be:

$$y_c = C_1 e^{-t} + C_2 e^{-2t}$$

Example 5 (y_p)

$$y''(t) + 3y' + 2y = 24, \quad y(0) = 10 \quad y'(0) = 0$$

$$y_c = C_1 e^{-t} + C_2 e^{-2t}$$

Once the complementary solution form is found, look at the form of f(t). In this case it is a constant (24), so the particular solution will be of the form, $y_p = A$

Example 5 (y_p)

$$y''(t) + 3y' + 2y = 24, \quad y(0) = 10 \quad y'(0) = 0$$

$$y_c = C_1 e^{-t} + C_2 e^{-2t}$$

$y_p = A$ Both y_p' and $y_p'' = 0$

Substituting into the DE $\Rightarrow$

$$\begin{aligned} y''(t) + 3y' + 2y &= 24 \\ 0 + 0 + 2y &= 24 \\ 2(A) &= 24 \\ A &= 12 \end{aligned}$$

$$y = y_c + y_p = C_1 e^{-t} + C_2 e^{-2t} + 12$$

Example 5 (cont)

$$y = y_c + y_p = C_1 e^{-t} + C_2 e^{-2t} + 12$$

Applying the two boundary conditions we get

$$y = C_1 e^{-t} + C_2 e^{-2t} + 12$$
$$y(0) = 10 = C_1 e^{-t} + C_2 e^{-2t} + 12$$
$$10 = C_1 e^{-(0)} + C_2 e^{-2(0)} + 12$$
$$-2 = C_1 + C_2$$

$$y' = -C_1 e^{-t} + -2C_2 e^{-2t}$$
$$y'(0) = 0 = -C_1 e^{-(0)} + -2C_2 e^{-2(0)}$$
$$0 = -C_1 e^{-(0)} + -2C_2 e^{-2(0)}$$
$$0 = -C_1 - 2C_2$$

Example 5 (cont)

$$y = y_c + y_p = C_1 e^{-t} + C_2 e^{-2t} + 12$$

Solving at the boundary conditions

$$\begin{aligned} -2 &= C_1 + C_2 \\ 0 &= -C_1 - 2C_2 \\ \hline -2 &= -C_2 \\ C_2 &= \boxed{2} \end{aligned}$$

and

$$\begin{aligned} -2 &= C_1 + C_2 \\ -2 &= C_1 + 2 \\ C_1 &= -4 \end{aligned}$$

Making the substitutions, we get,

$$\boxed{y = -4e^{-t} + 2e^{-2t} + 12}$$

Example 6

solve :

$$y''(t) + 2y' + 5y = 0,$$

Boundary conditions: $y(0) = 0$, $y'(0) = 10$

The DE is homogeneous so there will only be a complementary solutions, y_c.

$$p^2 + 2p + 5 = 0 \quad \text{roots} \Rightarrow P = 1 \pm j2$$

Example 6 (continued)

$$y''(t) + 2y' + 5y = 0, \quad y(0) = 0 \quad y'(0) = 10$$

$$\text{roots} \Rightarrow P = 1 \pm j2$$

$$y_c = y = C'_1 e^{(-1+j2)t} + C'_2 e^{(-1-j2)t}$$

$$y_c = y = C_1 e^{-1t} \sin 2t + C_2 e^{-1t} \cos 2t$$

$$y(0) = 0 = C_1 e^{-1t} \sin 2t + C_2 e^{-1t} \cos 2t$$

$$0 = C_1 e^{-1(0)} \sin 2(0) + C_2 e^{-1(0)} \cos 2(0)$$

$$0 = C_1 (1)(0) + C_2 (1)(1)$$

$$\boxed{C_2 = 0}$$

Example 6 (cont)

$$y = C_1 e^{-t} \sin 2t + C_2 e^{-t} \cos 2t$$

$$y' = 2C_1 e^{-t} \cos 2t - C_1 e^{-t} \sin 2t + \left[-2C_2 e^{-t} \sin 2t - C_2 e^{-t} \cos 2t \right]$$

applying the boundary condition

$$y'(0) = 10 = 2C_1 e^{-(0)} \cos 2(0) - C_1 e^{-(0)} \sin 2(0) + \left[-2C_2 e^{-(0)} \sin 2(0) - C_2 e^{-(0)} \cos 2(0) \right]$$

Example 6 (cont)

$$y'(0) = 10 = 2C_1 e^{-(0)} \cos 2(0) - C_1 e^{-(0)} \sin 2(0) + \left[-2C_2 e^{-(0)} \sin 2(0) - C_2 e^{-(0)} \cos 2(0)\right]$$

$$10 = 2C_1(1)(1) - C_1(1)(0) + \left[-2C_2(1)(0) - C_2(1)(1)\right]$$

$$10 = 2C_1 - C_2$$

$$10 = 2C_1 - 0 \qquad \Leftarrow \text{(substituting } C_2 = 0 \text{ from previous slide)}$$

$$C_1 = 5$$

$$y(t) = 5e^{-t} \sin 2t + 0e^{-t} \cos 2t$$

$$y(t) = \boxed{5e^{-t} \sin 2t}$$

Example 7

Solve : $y'' + 3y' + 2y = 24$

Boundary conditions
$$\boxed{\begin{aligned} y(0) &= 0 \\ y'(0) &= 10 \end{aligned}}$$

The $y_p = A$ so $y' \text{ and } y'' = 0$

$$0 + 0 + 2y = 24$$

$$2(A) = 24$$

$$A = 12$$

Example 7 (continued)

$A = 12$

$$y_c = C_1 e^{-t} + C_2 e^{-2t}$$

$$y = C_1 e^{-t} + C_2 e^{-2t} + 12$$

Applying the boundary conditions,

$$y(0) = 0 = C_1 e^{-t} + C_2 e^{-2t} + 12$$

$$-12 = C_1 e^{-(0)} + C_2 e^{-2(0)}$$

$$\boxed{-12 = C_1 + C_2}$$

$$y' = -C_1 e^{-t} - 2C_2 e^{-2t}$$

$$y'(0) = 10 = -C_1 e^{-(0)} - 2C_2 e^{-2(0)}$$

$$\boxed{10 = -C_1 - 2C_2}$$

$C_1 = -14$ and $C_2 = 2$

$$\boxed{y = -14e^{-t} + 2e^{-2t} + 12}$$

Example 8

$$\text{Solve}: y'' + 3y' + 2y = 24e^{-4t}$$

Boundary conditions

$$y(0) = 10$$
$$y'(0) = 5$$

$$y_p = Ae^{-4t} \quad y' = -4Ae^{-4t} \quad y'' = 16Ae^{-4t}$$

Substituting we get,

$$24e^{-4t} = 16Ae^{-4t} + (3)\left(-4Ae^{-4t}\right) + 2Ae^{-4t}$$

$$24 = 16A - 12A + 2A$$

$$24 = 6A \Rightarrow \boxed{A = 4}$$

Example 8 (continued)

$$y_c = C_1 e^{-t} + C_2 e^{-2t} + y_p$$

$$y_c = C_1 e^{-t} + C_2 e^{-2t} + 4e^{-4t}$$

Applying the boundary conditions,

$$y(0) = 10 = C_1 e^{-(0)} + C_2 e^{-2(0)} + 4e^{-4(0)}$$

$$10 = C_1 + C_2 + 4 \Rightarrow \boxed{6 = C_1 + C_2}$$

$$y'_c = -C_1 e^{-t} + -2C_2 e^{-2t} + -16e^{-4t}$$

$$y'(0) = 5 = -C_1 e^{-(0)} + -2C_2 e^{-2(0)} + -16e^{-4(0)}$$

$$5 = -C_1 - 2C_2 - 16 \Rightarrow \boxed{21 = -C_1 - 2C_2}$$

Example 8 (continued)

Solving simultaineously,

$$\begin{aligned} 6 &= C_1 + C_2 \\ 21 &= -C_1 - 2C_2 \\ \hline 27 &= \quad\; - C_2 \end{aligned}$$

$$C_2 = \boxed{-27}$$

Substituting, we get,

$$6 = C_1 - 27$$

$$C_1 = \boxed{33}$$

$$\boxed{y = 33e^{-t} - 27e^{-2t} + 4e^{-4t}}$$

Additional Drill Problems

1)	$y' + 4y = 0$ $\quad \left[ans: 6e^{-4t} \right]$	$y(0) = 6$
2)	$y' + 4y = 40$ $\quad \left[ans: 10 - 4e^{-4t} \right]$	$y(0) = 6$
3)	$y' + 4y = 40\sin 3t$ $\left[ans: 4.8e^{-4t} + 6.4\sin 3t - 4.8\cos 3t \right.$ $\left. or: 4.8e^{-4t} + 8\sin(3t - 36.87^\circ) \right]$	$y(0) = 0$
4)	$y'' + 4y' + 3y = 0 \left[ans: 6e^{-t} - 6e^{-3t} \right]$	$y(0) = 0$ $y'(0) = 12$
5)	$y'' + 4y' + 3y = 30 \left[ans: -9e^{-t} - e^{-3t} + 10 \right]$	$y(0) = 0$ $y'(0) = 0$
6)	$y'' + 4y' + 3y = 12e^{-2t} \left[ans: -18e^{-t} - 6e^{-3t} - 12e^{-2t} \right]$	$y(0) = 0$ $y'(0) = 0$

Separable Equations

Some differential equations can be separated with all the x terms on one side and all the y terms on the other side. Then the equation can be solved by integrating both sides.

Example 9

$$0 = y' + 3\left(2y - \sin x\right) - \left(x\left(\sin x\right) + 6y\right)$$

$$0 = y' + 6y - 3\sin x + \left(-x\sin x\right) - 6y$$

$$0 = y' + \cancel{6y} - \cancel{6y} - 3\sin x - x\sin x$$

$$0 = y' - 3\sin x - x\sin x$$

$$y' = 3\sin x + x\sin x$$

Example 9 (continued)

$$y' = 3\sin x + x\sin x$$

$$\frac{dy}{dx} = 3\sin x + x\sin x$$

$$\int dy = \int \left(3\sin x + x\sin x\right) dx$$

$$y = 3\int \sin x dx + \int \left(x\sin x\right) dx$$

$$\boxed{y = -3\cos x + \sin x - x\cos x + C}$$