FE Math Review
and
EEET 300 Laplace Review

LAPLACE TRANSFORMS

Laplace Example 1

$$\mathcal{L}(s) = \frac{20}{s(s+10)}$$

What is the partial fraction expansion of $\mathcal{L}(s)$?

$$\frac{20}{s(s+10)} = \frac{A}{s} + \frac{C}{(s+10)}$$

$$A = \frac{20}{(s+10)}\bigg|_{s=0} = \frac{20}{(0+10)} = 2$$

$$C = \frac{20}{s}\bigg|_{s=-10} = \frac{20}{-10} = -2$$

$$\frac{20}{s(s+10)} = \boxed{\frac{2}{s} - \frac{2}{(s+10)}}$$

Laplace Example 2

$$\mathcal{L}(s) = \frac{20}{s(s+10)} = \frac{2}{s} - \frac{2}{(s+10)}$$

What is the inverse Laplace transform of $\mathcal{L}(s)$?

$$\mathcal{L}^{-1}\left\{\frac{2}{s}\right\} = 2$$

$$\mathcal{L}^{-1}\left\{\frac{-2}{(s+10)}\right\} = -2e^{-10t}$$

$$\mathcal{L}^{-1}\left\{\frac{20}{s(s+10)}\right\} = \boxed{2 - 2e^{-10t}}$$

Laplace Example 3

$$\mathcal{L}(s) = \frac{s(s+10)}{(s+5)(s+15)}$$

Before going any further, you need to ensure that the order of the denominator is greater than the numerator.

$$\frac{s^2 + 10s}{(s^2 + 20s + 75)}$$ the order of the denominator is = to the order of the numerator

So, divide the numerator by the denominator.

$$\begin{array}{r|l} & \quad 1 \\ s^2 + 20s + 75 & \overline{s^2 + 10s + 0} \\ & -(s^2 + 20s + 75) \\ \hline & -10s - 75 \end{array} \Rightarrow \boxed{1 + \frac{-10s - 75}{(s+5)(s+15)}}$$

Example 3 (cont)

$$\mathcal{L}(s) = 1 + \frac{-10s - 75}{(s+5)(s+15)}$$

What is the partial fraction expansion of $\mathcal{L}(s)$?

$$1 + \frac{-10s - 75}{(s+5)(s+15)} = 1 + \frac{A}{(s+5)} + \frac{C}{(s+15)}$$

$$A = \frac{-10s - 75}{(s+15)}\Big|_{s=-5}$$

$$A = \frac{-10(-5) - 75}{(-5+15)}$$

$$A = \frac{50 - 75}{10} = \frac{-25}{10} = \boxed{-2.5}$$

Example 3 (cont)

$$1+\frac{-10s-75}{(s+5)(s+15)}=1+\frac{-2.5}{(s+5)}+\frac{C}{(s+15)}$$

$$C=\frac{-10s-75}{(s+5)}\Bigg|_{s=-15}$$

$$C=\frac{-10(-15)-75}{(-15+5)}$$

$$C=\frac{150-75}{-10}=\frac{75}{-10}=\boxed{-7.5}$$

$$\boxed{\mathcal{L}(s)=1+\frac{-2.5}{(s+5)}+\frac{-7.5}{(s+15)}}$$

Example 4

$$\mathcal{L}(s) = 1 + \frac{-2.5}{(s+5)} + \frac{-7.5}{(s+15)}$$

What is the inverse Laplace transform of $\mathcal{L}(s)$?

$$\boxed{1 - 2.5e^{-5t} - 7.5e^{-15t}}$$

Example 5

What is the inverse Laplace transform of

$$\mathcal{L}(s) = \frac{20s}{s^2 + 20s + 75}$$

The partial fraction expansion will be of the form

$$\frac{20s}{(s+5)(s+15)} = \frac{A}{(s+5)} + \frac{C}{(s+15)}$$

Example 5 (continued)

$$\frac{20s}{(s+5)(s+15)} = \frac{A}{(s+5)} + \frac{C}{(s+15)}$$

$$A = \frac{20s}{(s+15)}\Big|_{s=-5} = \frac{20(-5)}{(-5+15)} = \frac{-100}{10} = \boxed{-10}$$

$$C = \frac{20s}{(s+5)}\Big|_{s=-15} = \frac{20(-15)}{(-15+5)} = \frac{-300}{-10} = \boxed{30}$$

$$\mathcal{L}^{-1}\left\{\frac{-10}{(s+5)} + \frac{30}{(s+15)}\right\} = \boxed{-10e^{-5t} + 30e^{-15t}}$$

Example 6

What is the Laplace transform of the step function f(t) below?

$$f(t) = u(t-1) + u(t-2)$$

$$\boxed{F(s) = \frac{e^{-1s}}{s} + \frac{e^{-2s}}{s}}$$

Example 7

What is the Laplace transform of $(\sin t)(\cos t)$?

First, the function has to be seperated into something which is represented on the transform table.

Using the double angle formula for $\sin 2t$ we get:

$$\sin 2t = 2\sin t\cos t \quad \therefore \quad (\sin t)(\cos t) = \frac{\sin 2t}{2}$$

$$\mathcal{L}\left((\sin t)(\cos t)\right) = \frac{1}{2}\mathcal{L}(\sin 2t) = \frac{1}{2}e^{-\alpha s}\left(\frac{\omega}{s^2+\omega^2}\right) \quad \text{where} \quad \begin{pmatrix}\alpha = 0\\ \omega = 2\end{pmatrix}$$

$$= \frac{1}{2}e^{-0s}\left(\frac{2}{s^2+2^2}\right) = \boxed{\frac{1}{2}\left(\frac{2}{s^2+4}\right)} = \boxed{\frac{1}{s^2+4}}$$

Laplace Transform of Derivatives

$$L(y') = sy - y(0)$$

$$L(y'') = s^2y - sy(0) - y'(0)$$

Example 8

Solve the following DE with Laplace Transforms.

$$0 = y'' + 4y' + 3y$$

Boundary conditions

$$y(0) = 3$$
$$y'(0) = 1$$

Use

$$\mathcal{L}(y') = sy - y(0)$$
$$\mathcal{L}(y'') = s^2 y - sy(0) - y'(0)$$

Example 8 (continued)

$$0 = y'' + 4y' + 3y$$
$$= s^2y - sy(0) - y'(0) + 4\left[sy - y(0)\right] + 3y$$
$$= s^2y - sy(0) - y'(0) + 4sy - 4y(0) + 3y$$
$$= s^2y - s(3) - 1 + 4sy - 4(3) + 3y$$
$$s^2y + 4sy + 3y = s(3) + 1 + 12$$
$$s^2y + 4sy + 3y = 3s + 13$$
$$(s+1)(s+3)y = 3s + 13$$
$$y = \frac{3s + 13}{(s+1)(s+3)}$$

Example 8 (cont)

$$Y = \frac{3s + 13}{(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

$$A = \frac{3s + 13}{(s+3)}\Big|_{s=-1} = \frac{3(-1) + 13}{((-1) + 3)} = \frac{10}{2} = \boxed{5}$$

$$B = \frac{3s + 13}{(s+1)}\Big|_{s=-3} = \frac{3(-3) + 13}{((-3) + 1)} = \frac{4}{-2} = \boxed{-2}$$

$$\mathcal{L}(s) = \frac{5}{(s+1)} + \frac{-2}{(s+3)}$$

$$\mathcal{L}^{-1}(s) = \boxed{5e^{-1t} - 2e^{-3t}}$$