Math Examples

FE Review
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Math Example 1

- **Question 1 – 3** relate to the three vectors *A*, B, and *C* in the Cartesian coordinate system; the unit vectors i, j, and k are parallel to the x, y, and z axis respectively.

$$\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\mathbf{B} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$$

$$\mathbf{C} = \mathbf{i} - \mathbf{k}$$

Example 1: Q1

- The angle between vectors A and B is:

$\measuredangle$ between A and B

defined as $\cos^{-1}\left[\dfrac{A \bullet B}{|A||B|}\right]$

a.	$-180°$
b.	$0°$
c.	$-45°$
d.	$90°$
e.	$180°$

$$\frac{A \bullet B}{|A||B|} = \frac{2(4) + 3(-2) + 1(-2)}{\sqrt{2^2 + 3^2 + 1^2}\sqrt{4^2 + (-2)^2 + (-2)^2}}$$

$$= \frac{8 - 6 + -2}{\sqrt{4 + 9 + 1}\sqrt{16 + 4 + 4}}$$

$$= \frac{0}{\sqrt{4 + 9 + 1}\sqrt{16 + 4 + 4}} = 0 \Rightarrow \boxed{\cos^{-1}(0) = 90°}$$

Example 1: Q2

- The projection of ***A*** on ***C*** is of directed length:

a.	$\frac{-\sqrt{2}}{2}$
b.	$\frac{1}{\sqrt{14}}$
c.	0
d.	$\frac{-1}{\sqrt{14}}$
e.	$\frac{\sqrt{2}}{2}$

$$P = \frac{A \bullet C}{|C|} = \frac{A_x C_x + A_y C_y + A_z C_z}{\sqrt{1^2 + 0^2 + (-1)^2}}$$

$$= \frac{2(1) + 3(0) + 1(-1)}{\sqrt{1^2 + 0^2 + (-1)^2}} = \frac{2-1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

$\frac{1}{\sqrt{2}}$ is not a choice. Rationalize the answer

$\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}$ which is answer e.

Example 1: Q3

- The volume of the parallelpiped with sides **A**, **B**, and **C** is

a.	$\sqrt{12}$
b.	6
c.	12
d.	18
e.	24

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} 2 & 3 & 1 \\ 4 & -2 & -2 \\ 1 & 0 & -1 \end{vmatrix}$$

$$V = A \bullet (B \times C)$$

$$2\begin{vmatrix} -2 & -2 \\ 0 & -1 \end{vmatrix} - 4\begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix}$$

$$2\left[-2(-1) - 0\right] - 4\left[3(-1) - 0\right] + 1\left[3(-2) - 1(-2)\right]$$

$$2(2) - 4(-3) + 1(-6 + 2) = 4 + 12 - 4 = \boxed{12 \Rightarrow \text{ans} \quad c}$$

Math Example 2

- **Questions 4 – 7:** The position **x** in **km** of a train traveling on a straight track is given as a function of time **t** in hours by the following equation:

$$x = \frac{t^4}{4} - 4t^3 + 16t^2$$

- The train moves from **point P** to **point Q** and back to **point P** according to the equation above. The direction from point **P** to **Q** is positive.

Example 2: Q4

- What is the train's distance from starting point **P** at time **t = 4 hrs**?

a.	0 km
b.	4 km
c.	16 km
d.	36 km
e.	64 km

$$x = \frac{t^4}{4} - 4t^3 + 16t^2$$

$$x = \frac{(4\text{hrs})^4}{4} - 4(4\text{hrs})^3 + 16(4\text{hrs})^2$$

$$x = \frac{256}{4} - 4(64) + 16(16)$$

$$= 64 - 256 + 256 = \boxed{64\text{ km}}$$

Example 2: Q5

- What is the train's velocity at time t = 4 hrs?

a.	$-16\ {}^{km}/_{h}$
b.	$8\ {}^{km}/_{h}$
c.	$0\ {}^{km}/_{h}$
d.	$32\ {}^{km}/_{h}$
e.	$64\ {}^{km}/_{h}$

By definition, velocity is the first derivative of position with respect to time

$$= \frac{dx}{dt}\left(\frac{t^4}{4} - 4t^3 + 16t^2\right)$$

$$= \frac{4t^3}{4} - 4(3)t^2 + 16(2)t$$

$$= t^3 - 12t^2 + 32t = (4h)^3 - 12(4h)^2 + 32(4h)$$

$$= 64 - 12(16) + 128 = \boxed{0\ {}^{km}/_{h}} \Rightarrow \text{ans } C$$

Example 2: Q6

- **What is the train's acceleration at time t = 4 hrs?**

By definition, acceleration is the first derivative of velocity with respect to time

a.	$-16\ {km}/{h^2}$
b.	$0\ {km}/{h^2}$
c.	$12\ {km}/{h^2}$
d.	$16\ {km}/{h^2}$
e.	$32\ {km}/{h^2}$

$$\frac{d}{dt}\left(t^3 - 12t^2 + 32t\right) = 3t^2 - 12\left(2\right)t + 32$$

$$= 3t^2 - 24t + 32 = 3\left(4h\right)^2 - 24\left(4h\right) + 32$$

$$= 3\left(16\right) - 96 + 32 = 48 - 96 + 32 = -16\ {km}/{h^2} \Rightarrow \text{ans a.}$$

Acceleration is negative therefore train is slowing down

Example 2: Q7

- What is the distance between points P and Q?

- Ans: From the answers for the previous questions we know that the **velocity is zero** and the **acceleration is negative**. From this we know that the train is at point P and is turning around. Therefore the distance is **64 km** (from question 4.

a.	16km
b.	32km
c.	64km
d.	72km
e.	128km

Example 2: Q7 continued:

- If we were asked Q7 without already having answering Q4 – Q6, we would need to determine at what time V=0, which indicates when the train is reversing direction, which would either be at P or Q.

set V = 0 in $\frac{dx}{dt} = v(t) = t^3 - 12t^2 + 32^t$

$0 = t^3 - 12t^2 + 32^t = t\left(t^2 - 12t + 32\right) = t(t-4)(t-8)$

at t = 4 x = 64km as above

at t = 8 back at point P

Math Example 3

- **Questions 8 – 11** Under certain conditions the motion of an oscillating spring is described by the differential equation

$$\frac{d^2x}{dt^2} + 16x = 0$$

where **x is the displacement** in feet of the end of the spring, and **t is the time in seconds**. At **t = 0 seconds** the displacement is $\frac{1}{4}$ foot and the **velocity is 0 ft/sec**; i.e. **x(0)=1/4 and x'(0) = 0**.

Example 3: Q8

- What is the general solution of the system? (C_1 and C_2 are constants)

a.	$x = c_1 e^{-t} + c_2 e^{-3t}$
b.	$x = c_1 e^{-4t} + c_2 e^{-4t}$
c.	$x = c_1 \sin 4t$
d.	$x = c_1 \cos 4t$
e.	$x = c_1 \cos 4t + C_2 \sin 4t$

$$\frac{d^2x}{dt^2} + 16x = 0,$$

$$\text{characteristic eqn} \quad r^2 + 16 = 0, \quad r = \pm 4j$$

$\therefore$ solution will be of the form:

$$x = C_1 \cos(4t) + C_2 \sin(4t) \Rightarrow \text{ans E}$$

Example 3: Q9

- The solution that fits the initial conditions is

a.	$x = \frac{1}{4} e^{-4t}$
b.	$x = \frac{1}{3} \sin 4t$
c.	$x = \frac{1}{3} \sin 4t + \frac{1}{4} \cos 4t$
d.	$x = 4 \cos 4t$
e.	$x = \frac{1}{4} \cos 4t$

Boundary conditions:

$x(0) = \frac{1}{4}\ ,\ x'(0) = 0$

$$x = C_1 \cos 4t + C_2 \sin 4t$$

$$x' = -4C_1 \sin 4t + 4C_2 \cos 4t$$

$$x'(0) = 0 \Rightarrow 0$$

$$= -4C_1 \underbrace{\sin 0}_{0} + 4C_2 \underbrace{\cos 0}_{1}$$

$$0 = 4C_2 \therefore \boxed{C_2 = 0}$$

Example 3: Q9 continued

$$x = C_1 \cos 4t + C_2 \sin 4t$$

$$x(0) = \tfrac{1}{4} \Rightarrow \tfrac{1}{4}$$

$$= C_1 \underbrace{\cos 0}_{1} + C_2 \underbrace{\sin 0}_{0}$$

$$\boxed{C_1 = \tfrac{1}{4}}$$

so,

$$x = \tfrac{1}{4} \cos 4t \Rightarrow \text{ans E}$$

a.	$x = \tfrac{1}{4} e^{-4t}$
b.	$x = \tfrac{1}{3} \sin 4t$
c.	$x = \tfrac{1}{3} \sin 4t + \tfrac{1}{4} \cos 4t$
d.	$x = 4 \cos 4t$
e.	$x = \tfrac{1}{4} \cos 4t$

Example 3: Q10

- The amplitude of the motion is

a.	$\frac{1}{4}$ ft
b.	$\frac{1}{3}$ ft
c.	1 ft
d.	2 ft
e.	4 ft

The amplitude will be the maximum value of $x = \frac{1}{4}\cos 4t$. Since the maximum value of cos 4t is 1, the max value of the function is

$$\frac{1}{4}\cos 4t = \frac{1}{4}(1) = \frac{1}{4} \Rightarrow \text{ans } A$$

Example 3: Q11

- The period of the motion is

$$x = \tfrac{1}{4}\cos 4t$$

a.	$\pi/3$ sec
b.	$\pi/2$ sec
c.	π sec
d.	2π sec
f.	3π sec

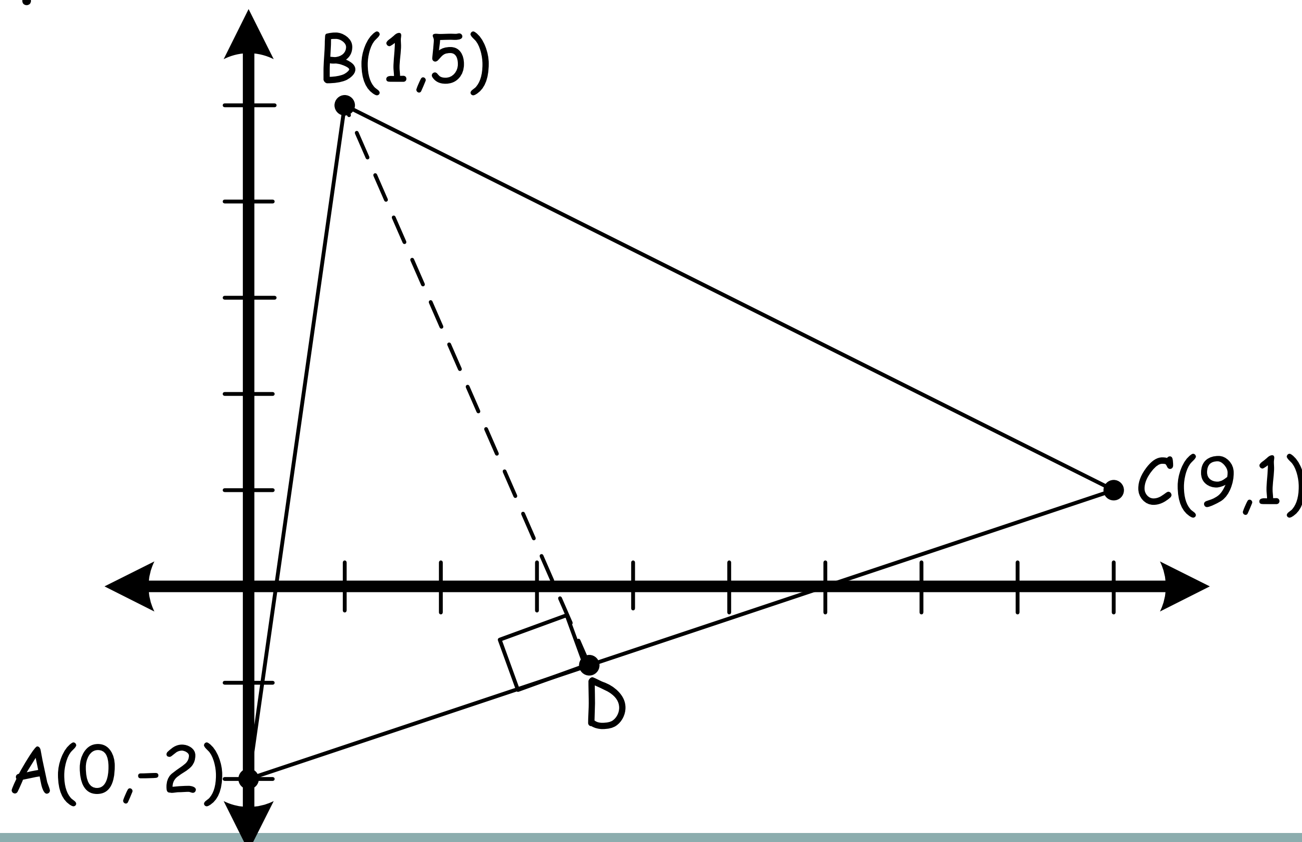
The natural frequency is $\sqrt{16} = 4\ \mathrm{rad/s}$

$$\text{Period} = \frac{1}{f}, \text{but } f = \frac{\omega}{2\pi} \text{ so,}$$

$$P = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \boxed{\frac{\pi}{2} \Rightarrow \text{ans B}}$$

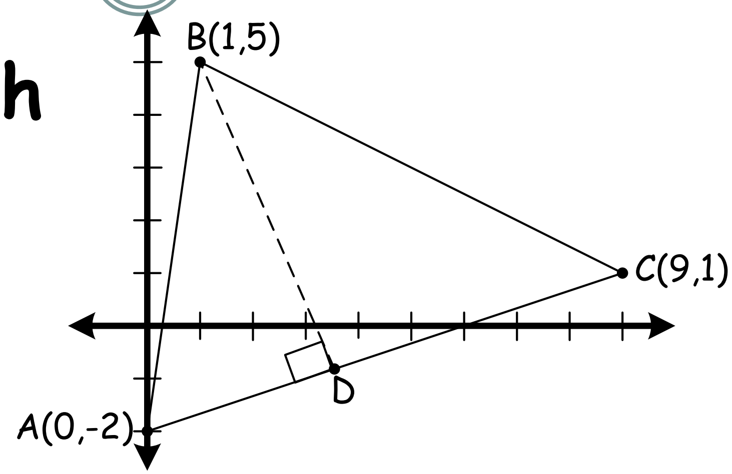
Example 4

- Questions 12-15: Triangle ABC has vertices as shown in the figure below. The line BD is perpendicular to the line AC.



Example 4: Q12

- **What is the length of line *AC***



a.	$2\sqrt{3}$
b.	$3\sqrt{10}$
c.	10
d.	11
e.	12

$$\overline{AC} = \sqrt{x^2 + y^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

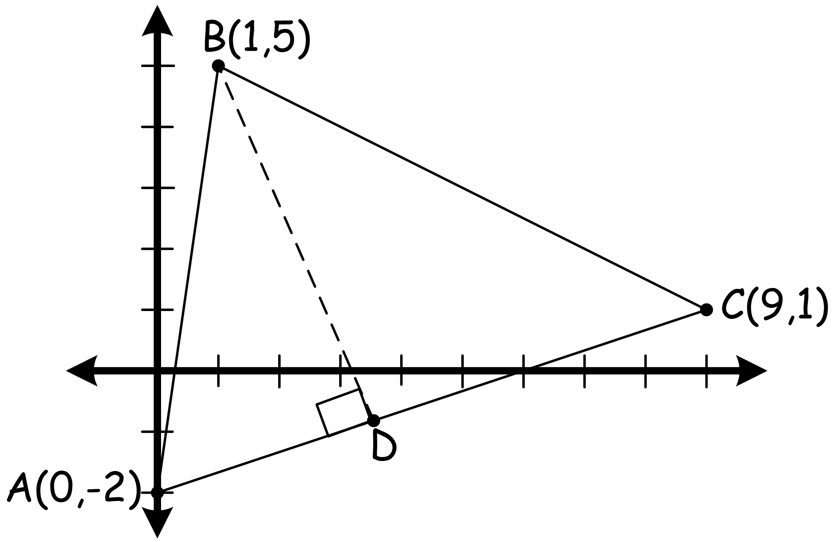
$$= \sqrt{(9 - 0)^2 + (1 - [-2])^2}$$

$$= \sqrt{(9)^2 + (3)^2} = \sqrt{81 + 9} = \sqrt{90} = \sqrt{9 \bullet 10} = \boxed{3\sqrt{10} \Rightarrow \text{ans B}}$$

Example 4: Q13

a.	$y = \frac{1}{3}x - 2$
b.	$y = -\frac{1}{3}x + 2$
c.	$x + 3y = 6$
d.	$x - 3y + 6 = 0$
e.	$x + y = 6$

- What is the equation of line AC



Point-to-point eqn: $\frac{y-y_1}{x-x_1}$

$$\frac{y_2-y_1}{x_2-x_1} = \frac{1-(-2)}{9-0} = \frac{3}{9} = \frac{1}{3} = \text{slope so,}$$

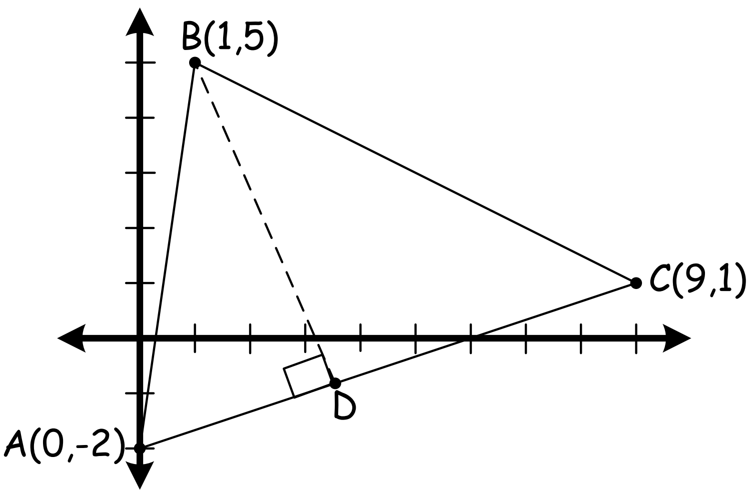
$$\frac{y-(-2)}{x-(0)} = \frac{1}{3}$$

$$\frac{y+2}{x} = \frac{1}{3}$$

$$y + 2 = \frac{1}{3}x \Rightarrow \boxed{y = \frac{1}{3}x - 2 \Rightarrow \text{ans } A}$$

Example 4: Q14

- **What is the slope of line BD?**



a.	-5
b.	-3
c.	$-\frac{1}{3}$
d.	$\frac{1}{3}$
e.	3

$\overline{BD}$ is perpendicular to $\overline{AC}$

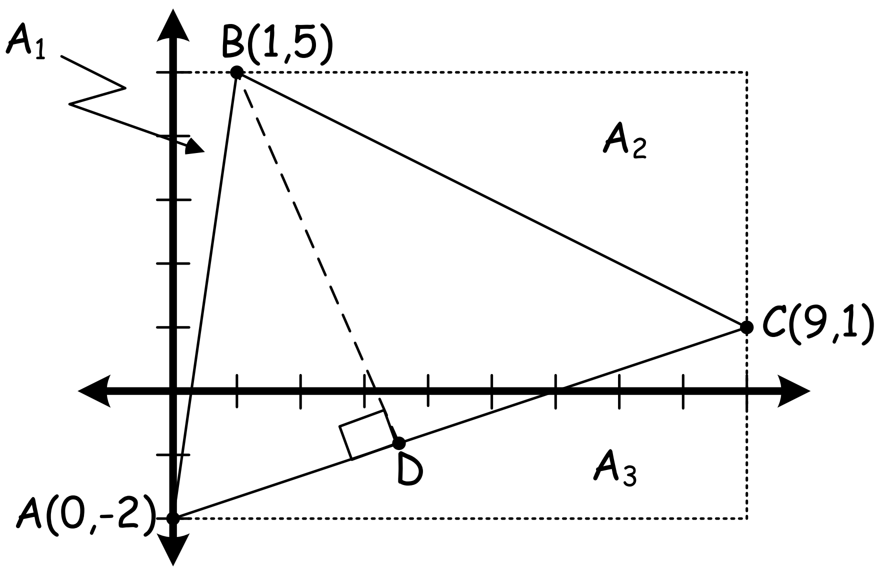
$$\text{so, } m_{BD} = -\frac{1}{m_{AC}}$$

$$= -\frac{1}{\frac{1}{3}} = \boxed{-3 \Rightarrow \text{ans B}}$$

Example 4: Q15

- **What is the area of the triangle ABC?**

a.	$5\sqrt{10}$
b.	$6\sqrt{10}$
c.	26
d.	30
e.	32



There are several ways to solve this. The easiest way is to box the triangle, find the area of the rectangle and subtract out the outside triangles

$$A_T = A_R - A_1 - A_2 - A_3$$

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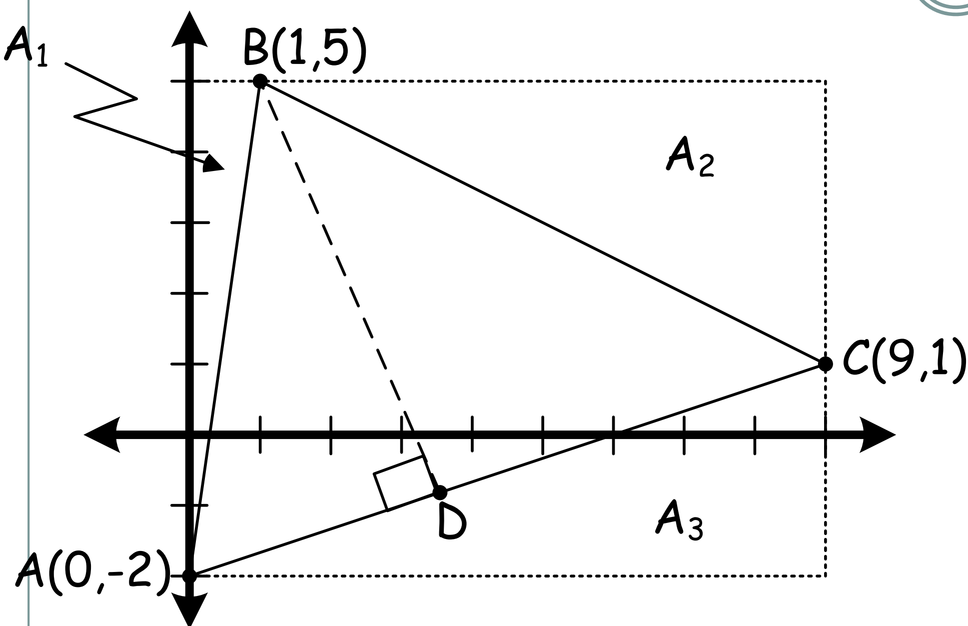
Example 4: Q15(cont)



$$AR = xy = (9-0)(5-[-2])$$
$$= (9)(7) = 65$$
$$A_1 = \frac{1}{2}xy = \frac{1}{2}(1-0)(5-[-2])$$
$$= \frac{1}{2}(1)(7) = \frac{7}{2}$$
$$A_2 = \frac{1}{2}(9-1)(5-1)$$
$$= \frac{1}{2}(8)(4) = \frac{32}{2} = 16$$
$$A_3 = \frac{1}{2}(9-0)(1-[-2])$$
$$= \frac{1}{2}(9)(3) = \frac{27}{2}$$
$$A_T = 65 - \frac{7}{2} - 16 - \frac{27}{2}$$
$$= 65 - 16 - \frac{34}{2} = 49 - 17 = 32$$

Morning Session Typical Questions Questions M1 - M10

Question M1

An equation of the straight line through the point (6,2) with a slope of 3 is

a.	$y = 3x + 16$
b.	$y = 3x + 20$
c.	$y = 3x - 16$
d.	$x = 3y - 16$
e.	$x = y/3 + 16$

$$m = \frac{\text{rise}}{\text{run}} = \frac{y - y_1}{x - x_1}$$

$$\text{point}\,(6,2) \Rightarrow 3 = \frac{y-2}{x-6}$$

$$y - 2 = 3(x - 6)$$

$$y = 3x - 18 + 2$$

$$y = \boxed{3x - 16}$$

Question M2

Consider the function of x is equal to the determinant shown below.

$$f(x) = \begin{vmatrix} x & x^2 \\ x^4 & x^3 \end{vmatrix}$$

a.	$3x^2 - 8x^4$
b.	$3x^2 + 7x^6$
c.	$4x^3 - 6x^5$
d.	$x^4 - x^6$
e.	$3x^4 - 5x^6$

The first derivative f'(x) of this function with respect to x is equal to:

$$f(x) = \begin{vmatrix} x & x^2 \\ x^4 & x^3 \end{vmatrix}$$

$$= (x * x^3) - (x^2 * x^4)$$

$$= x^4 - x^6$$

$$f'(x) = \boxed{4x^3 - 6x^5 \Rightarrow \text{ans c}}$$

Question M3

If the functional form of a curve is known, differentiation can be used to determine all of the following EXCEPT the:

a. Slope of the curve
b. Concavity of the curve
c. Location of inflection points on the curve
d. Number of inflection points on the curve
e. Area under the curve between certain bounds

The answer is obviously e. Area is found thru integration, not differentiation!

Question M4

If f' denotes the derivative of a function f, then f'(x) is defined by

a.	$\lim_{\Delta y \to \infty} \frac{\Delta x}{\Delta y}$
b.	$\lim_{\Delta y \to 0} \frac{\Delta y}{\Delta x}$
c.	$\lim_{\Delta x \to 0} \frac{f(x) - f(x - \Delta x)}{\Delta x}$
d.	$\lim_{\Delta x \to 0} \frac{f(x - \Delta x) - f(x)}{\Delta x}$
e.	$\lim_{\Delta y \to 0} \frac{f(x - \Delta x) - f(x)}{\Delta y}$

by definition, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x - \Delta x) - f(x)}{\Delta x} \Rightarrow$ ans d.

Question M5

One wishes to estimate the mean, M, of a population from a sample size, n, drawn from the population. For the sample, the mean is $\bar{x}$ and the standard deviation is 's'. The probable accuracy of the estimate improves with increase in:

a.	M
b.	n
c.	$\bar{x}$
d.	s
e.	none of the above

Answer : the larger the sample size, the greater the accuracy, therefore, the answer is b.

Question M6

$dy/dt + 5y = 0;$ $y(0) = 1$

Which of the following is the general solution to the differential equation and boundary condition shown above?

a.	e^{3t}
b.	e^{-3t}
c.	$e^{\sqrt{-5t}}$
d.	$5e^{-5t}$
e.	$5e^{-5t} - 5e^{-3t}$

$$\frac{dy}{dt} + 5y = 0, \quad y(0) = 1$$

$$\frac{dy}{dt} = -5y$$

$$\int \frac{dy}{y} = \int -5dt$$

Question M6 (continued)

$$\int \frac{dy}{y} = \int -5dt$$

$$\ln y = -5t + C$$

$$y = e^{-5t+C} = e^{-5t}e^{C} = ke^{-5t}$$

applying the boundary condition $y(0) = 1$

$$1 = ke^{-5(0)} = K = 1$$

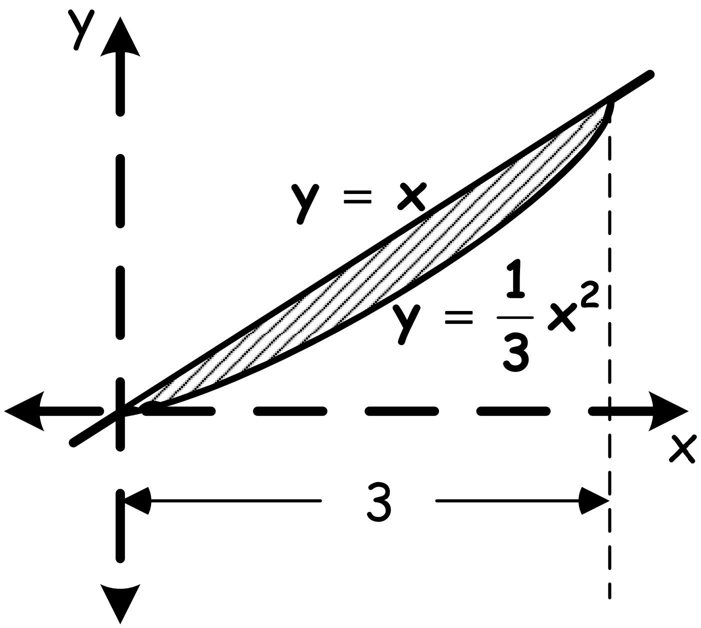
$$\boxed{y = e^{-5t}} \Rightarrow \text{ans b.}$$

Question M8

Which of the following expressions gives the distance from the y-axis to the centroid of the shaded area below?

$$\bar{x} = \frac{M_y}{A} = \frac{\int x\,dA}{A} = \frac{\int_0^3 x\left(f_1(x) - f_2(x)\right)dx}{\int_0^3 \left(f_1(x) - f_2(x)\right)dx}$$

$$= \frac{\int_0^3 x\left(x - \frac{1}{3}x^2\right)dx}{\int_0^3 \left(x - \frac{1}{3}x^2\right)dx} = \frac{\int_0^3 \left(x^2 - \frac{1}{3}x^3\right)dx}{\int_0^3 \left(x - \frac{1}{3}x^2\right)dx}$$



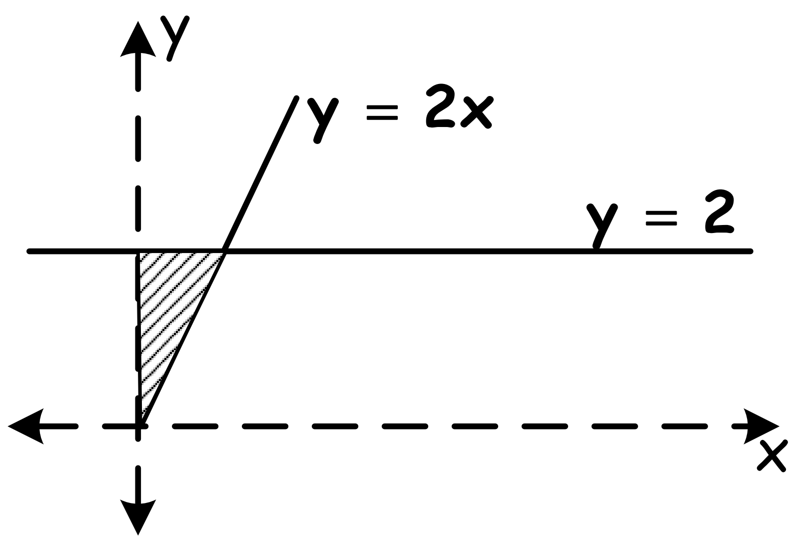
Question M8

$$= \frac{\left.\left(\frac{x^3}{3} - \frac{x^4}{3(4)}\right)\right|_0^3}{\left.\left(\frac{x^2}{2} - \frac{x^3}{3(3)}\right)\right|_0^3} = \frac{\left(\frac{3^3}{3} - \frac{3^4}{3(4)}\right) - 0}{\left(\frac{3^2}{2} - \frac{3^3}{3^2}\right) - 0}$$

$$= \frac{\left(\frac{27}{3} - \frac{3^4}{3(4)}\right)}{\left(\frac{9}{2} - 3\right)} = \frac{\left(9 - \frac{27}{4}\right)}{\left(\frac{9}{2} - \frac{6}{2}\right)} = \frac{\left(\frac{36}{4} - \frac{27}{4}\right)}{\frac{3}{2}}$$

$$= \frac{9/4}{3/2} = \frac{9}{4} \bullet \frac{2}{3} = \boxed{\frac{3}{2}}$$

Example 5: Questions M9 – M11 pertain to the figure below. The shaded region is bounded by the lines $x = 0$, $y = 2$, and $y = 2x$.



Consider the following five Quantities related to the Shaded area.

a. The area
b. The 1st moment of the area about the x-axis
c. The 1st moment of the area about the y-axis
d. The moment of inertia of area about the x-axis.
e. The moment of inertia of area about the y-axis.

For each numbered integral in the following 3 questions, select the single quantity of a.-e. which is generally found with that integral. One quantity (a.-e.) may be used once, more than once, or not at all.

Question M9

Solve $\int_0^1 x^2 (2 - 2x)\, dx$

$$\int_0^1 x^2 (2 - 2x)\, dx = \int_0^1 x^2 \left(f_2(x) - f_1(x)\right) dx$$

e. The moment of inertial of area about the y-axis.

Question M10

Solve $\int_0^1 (2 - 2x)\,dx$

$$\int_0^1 (2 - 2x)\,dx = \int_0^1 \left(f_2(x) - f_1(x)\right)dx$$

a. The area.

Question M11

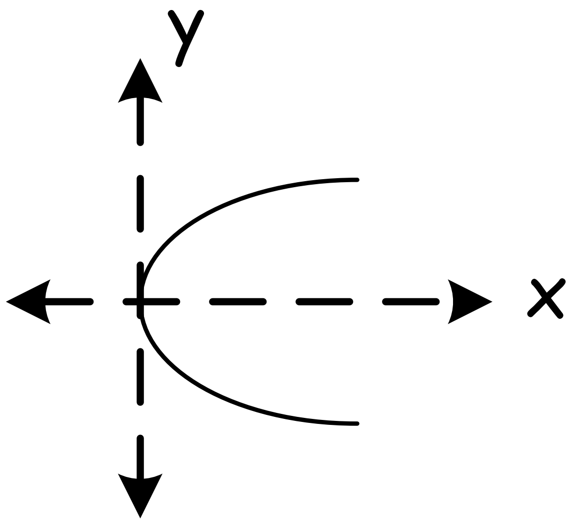
Solve $\int_0^2 \int_0^1 dx\ dy$

$\int_0^2 \int_0^1 dx\ dy$ = sum of areas dx dy

a. The area.

M12: The general equation of a second degree waveform is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$



a. B, C, D, and F = 0; A & E pos.
b. B, C, D, and F = 0; A neg. & E pos.
c. A, B, E, and F = 0; C & D pos,
d. A, B, E, and F = 0; D neg & C pos
e. A, D, E, and F = 0; B neg & C pos.

Waveform is a "Parabola" open to the right with the vertex at the origin (0,0). A parabola of this type would have an equation of the form $x = ny^2$ or more generally,

$Dx + Cy^2 = 0$ therefore, A, B, E, and F = 0

Since x must be positive, when y is either positive or negative, D OR C must be negative.