Using Complex Numbers

Algebra of Complex Numbers

Complex numbers may be designated in **Rectangular form, Polar form, or exponential form.**

- **Rectangular form**, a complex number is written in terms of its **real** and **imaginary** components:

$$Z = R \pm jX, \quad \text{where}$$
$$R = \text{Real component of the number}$$
$$X = \text{Imaginary component of the number}$$
$$j = \sqrt{-1} \qquad (\text{Engineering form})$$
$$\text{or}$$
$$i = \sqrt{-1} \qquad (\text{Math and Physics form})$$

- **Polar form**, a complex number is written in terms of a phasors **magnitude** and its **phase angle**:

$$\overline{Z} = A\measuredangle\theta, \quad \text{where}$$
$$A = \sqrt{R^2 + X^2} \qquad (\text{also known as modulus of Z})$$
$$\theta = \tan^{-1}\left(\frac{\text{Imaginary}}{\text{Real}}\right) = \tan^{-1}\left(\frac{X}{R}\right)$$
$$\text{Real} = A\cos\theta$$
$$\text{Imaginary} = A\sin\theta$$

- **Exponential form** is another way to represent polar form numbers: $A\measuredangle\theta \Rightarrow Ae^{j\theta}$

Complex numbers are added and subtracted in **rectangular form**.

if

$$Z_1 = R_1 + jX_1 = A_1(\cos\theta_1 + j\sin\theta_1) = A_1\measuredangle\theta_1$$
$$Z_2 = R_2 + jX_2 = A_2(\cos\theta_2 + j\sin\theta_2) = A_2\measuredangle\theta_2$$

then

$$Z_1 + Z_2 = (R_1 + R_2) + j(X_1 + X_2)$$
$$Z_1 - Z_2 = (R_1 - R_2) + j(X_1 - X_2)$$

While complex numbers are multiplied and divided in **polar form**.

$$Z_1 \bullet Z_2 = (A_1 \bullet A_2)\measuredangle(\theta_1 + \theta_2)$$
$$\frac{Z_1}{Z_2} = \left(\frac{A_1}{A_2}\right)\measuredangle(\theta_1 - \theta_2)$$

Complex Number Examples

Problem 1: Express $18\measuredangle 33.7^\circ$ in **rectangular form**.

$$= 18\cos 33.7^\circ + j18\sin 33.7^\circ = \boxed{14.98 + j9.987}$$

Problem 2: Express $14 + j20$ in **polar form**.

$$= \sqrt{14^2 + 20^2}\ \tan^{-1}\left(\frac{20}{14}\right) = \boxed{24.41\measuredangle 55.01^\circ} \quad \left[\text{quadrant 1}\right]$$

Problem 3: Express $48\measuredangle -45^\circ$ in **rectangular form**.

$$= 48\cos 45^\circ + j48\sin 45^\circ = \boxed{33.94 - j33.94} \qquad \left[\text{Quadrant 4}\right]$$

Problem 4: Perform: $\left(22\measuredangle -40^\circ\right) \bullet \left(11\measuredangle -60^\circ\right)$

$$= \left(22 \bullet 11\right)\measuredangle\left(-40^\circ + -60^\circ\right) = \boxed{242\measuredangle -100^\circ}$$

Problem 5: Perform: $\left(22 + j9\right) - \left(-11 + j6\right)$

$$= \left[22 - \left(-11\right)\right] + j\left[9 - 6\right] = \boxed{33 + j3}$$

Problem 6: Perform: $\dfrac{18 + j4}{6 - j8}$

$$= \frac{\sqrt{18^2 + 4^2}\measuredangle \tan^{-1}\left(\frac{4}{18}\right)\left[\text{quadrant 1}\right]}{\sqrt{6^2 + 8^2}\measuredangle \tan^{-1}\left(\frac{-8}{6}\right)\left[\text{quadrant 4}\right]}$$

$$= \frac{18.44\measuredangle 12.53^\circ}{10\measuredangle -53.13^\circ} = \frac{18.44}{10}\measuredangle\left(12.53^\circ - \left(-53.13^\circ\right)\right) = \boxed{1.844\measuredangle 65.66^\circ}$$

Problem 7: Perform the following with the result in **polar format**: $48\measuredangle 60^\circ + 80\measuredangle -135^\circ$

$$48\measuredangle 60^\circ + 80\measuredangle -135^\circ$$

$$48\cos 60 + j48\sin 60 + 80\cos\left(-135\right) + j80\sin\left(-135\right)$$

$$24 + j41.57 + \left(-56.57\right) + j\left(-56.57\right)$$

$$= 24 - 56.57 + j\left(41.57 - 56.57\right)$$

$$= -32.57 - j15 = \sqrt{32.57^2 + 15^2}\ \tan^{-1}\underbrace{\left(\frac{-15}{-32.57}\right)}_{\text{quadrant 3}}$$

$$= 35.86\measuredangle\left(24.73 - 180\right) = \boxed{35.86\measuredangle\left(-155.3^\circ\right)}$$

Phasor Transforms of Sinusoids

$$V_{peak}\cos(\omega t + \theta) = \frac{V_{peak}}{\sqrt{2}}\measuredangle\theta = \bar{V}$$

$$I_{peak}\cos(\omega t + \theta) = \frac{I_{peak}}{\sqrt{2}}\measuredangle\theta = \bar{I}$$

Problem 8: Express $v = 212\sin(\omega t + 45°)\,V$ in phasor notation.

$$v = 212\sin(\omega t + 45°)\,V = 212\cos(\omega t + 45° - 90°)\,V$$
$$= 212\cos(\omega t - 45°)\,V$$
$$\bar{v} = \frac{212}{\sqrt{2}}\measuredangle -45° = \boxed{149.9\measuredangle -45°V}$$

Problem 9: Express $v = 141.4\cos(\omega t - 90°)\,V$ in phasor notation.

$$v = 141.4\cos(\omega t - 90°)\,V$$
$$\bar{v} = \frac{141.4}{\sqrt{2}}\measuredangle -90° = \boxed{99.98\measuredangle -90°V}$$

Problem 10: Express $v = 38\cos(\omega t + 15°)\,V$ in phasor notation.

$$v = 38\cos(\omega t + 15°)\,V$$
$$\bar{v} = \frac{38}{\sqrt{2}}\measuredangle 15° = \boxed{26.87\measuredangle 15°V}$$

Problem 11: Express $v = 24\measuredangle 60°V$ as a sinusoid.

$$v = \left(\sqrt{2}\bullet 24\right)\cos(\omega t + 60°)\,V$$
$$v = \boxed{33.94\cos(\omega t + 60°)\,V}$$

Problem 12: Express $v = 10\measuredangle -45°V$ as a sinusoid.

$$v = \left(\sqrt{2}\bullet 10\right)\cos(\omega t - 45°)\,V$$
$$v = \boxed{14.144\cos(\omega t - 45°)\,V}$$

INDUCTORS and CAPACITORS in AC CIRCUITS
3 Things You Should Know

(1) In the simplest sense, **CAPACITORS** and **INDUCTORS** can be considered to be frequency-dependent resistors

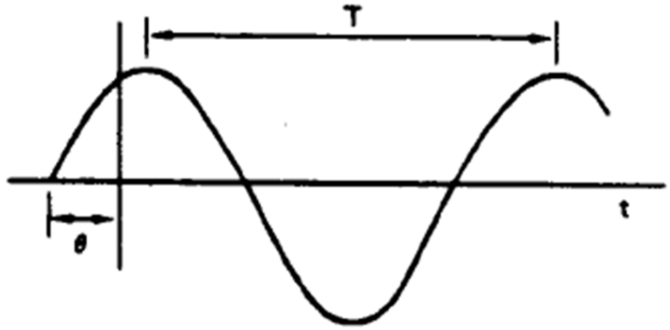
- **CIRCUIT IMPEDANCE** due to **RESISTORS (R)** is called **RESISTANCE**
- **CIRCUIT IMPEDANCE** due to **CAPACITORS (X_C)** and **INDUCTORS (X_L)** is called **REACTANCE**.

BOTH RESISTANCE and REACTANCE are MEASURED IN OHMS.

$$\boxed{X_C = \frac{1}{\omega C} \quad \text{and} \quad X_L = \omega L}$$

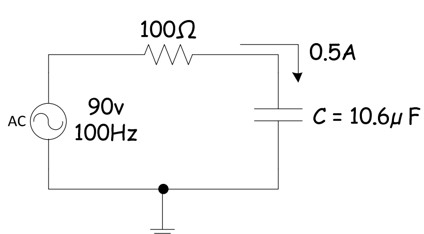
$$v(t) = V_m \sin(\omega t + \theta)$$

$$\boxed{\begin{aligned} f &= \frac{1}{T} = \frac{\omega}{2\pi} \\ \omega &= 2\pi f = \frac{2\pi}{T} \end{aligned}}$$



Sinusoidal Waveform with Phase Angle

Example: **Calculate the impedance given by the capacitor in the circuit below:**



Calculate $X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi(100hz)10.6\mu f} = \boxed{150\Omega}$

$Z_C = \boxed{150\Omega\measuredangle + 90^\circ}$

(2) Even though RESISTANCE and REACTANCE are both measured in OHMS, you can't just add them together to get impedance!

IMPEDANCE $\neq$ RESISTANCE + REACTANCE

a. This is because **RESISTANCE** is a **REAL #** and **REACTANCE** is an **IMAGINARY #**. Together they make up a **COMPLEX #**.

$$Z_{magnitude} = R + j\left(X_L - X_C\right) = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

$Z_R = R$ $Z_L = +jX_L$ $Z_C = -jX_C$

10Ω	10Ω	10Ω
10Ω	+j10Ω	-j10Ω

(3) CAPACITORS and INDUCTORS affect the phase relationship between current and voltage:

a. The phase of the impedance of a **capacitor** is **-90 degrees**
b. The phase of the impedance of an **inductor** is **+90 degrees**

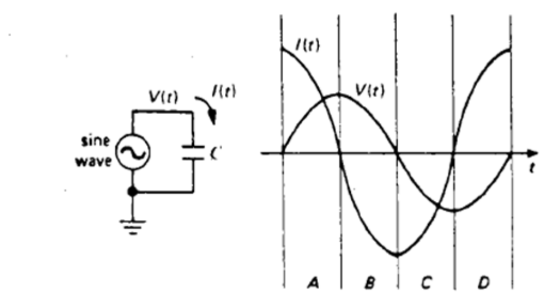


FIGURE 1

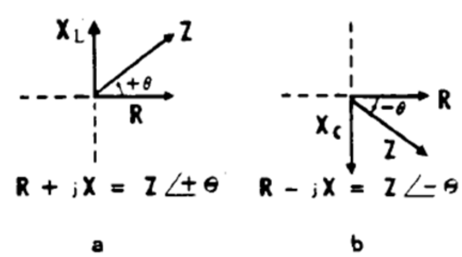


FIGURE 2

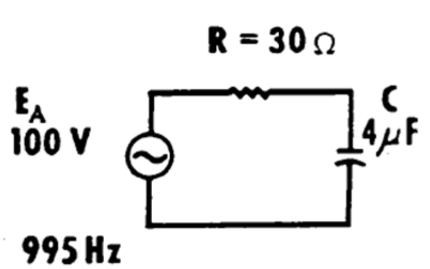


FIGURE 3

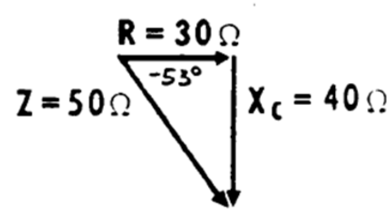
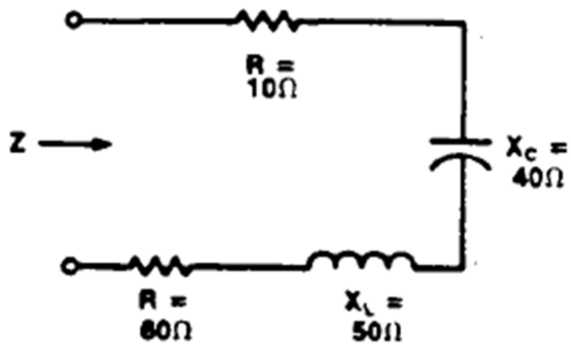


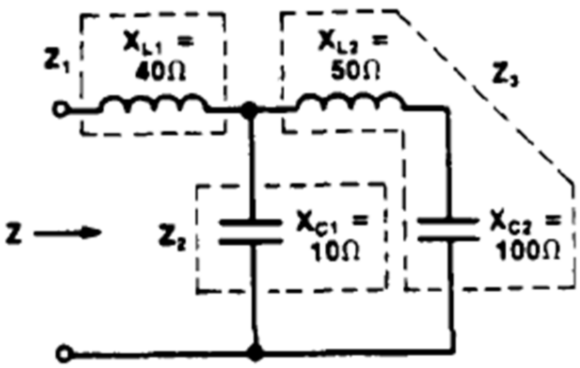
FIGURE 4

PROBLEMS:

1. Given the following **RLC** circuits, express the impedance of the circuit in both **rectangular** and **polar form**.



$$\begin{aligned} Z &= 10\Omega - j40\Omega + j50\Omega + 60\Omega \\ &= \boxed{70\Omega + j10\Omega} \\ &= \boxed{70.71\measuredangle 8.13^\circ} \end{aligned}$$



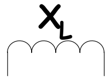
$$\begin{aligned} Z_3 &= j50 - j100 = -j50 = 50\measuredangle -90^\circ \\ Z_2 &= -j10 = 10\measuredangle -90^\circ \\ Z_1 &= +j40\Omega = 40\measuredangle 90^\circ \\ Z_T &= Z_1 + Z_2 \,||\, Z_3 \\ &= +j40\Omega + \frac{(10\measuredangle -90^\circ)(50\measuredangle -90^\circ)}{-j10 - j50} \\ &= +j40\Omega + \frac{500\measuredangle -180^\circ}{60\measuredangle -90^\circ} \\ &= +j40\Omega - j8.333 = \boxed{j31.67} = \boxed{31.67\measuredangle 90^\circ} \end{aligned}$$

2. Express the **resistance** or the **reactance** for the following elements in **rectangular and in polar form**:

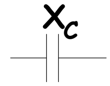
R

10Ω

Rectangular = $\boxed{10\Omega}$
Polar = $\boxed{10\measuredangle 0^\circ}$

X_L

50Ω

Rectangular = $\boxed{+j50\Omega}$
Polar = $\boxed{50\measuredangle 90^\circ}$

X_C

40Ω

Rectangular = $\boxed{-j40\Omega}$
Polar = $\boxed{40\measuredangle -90^\circ}$

PROBLEMS:

Example 1: A **series** circuit of **R = 10 ohms** and ***C* = 40μF** has an **applied voltage** of:
$v(t) = 500\cos(2500t - 20°)$. **Find the current i(t).**

$$X_c = \frac{1}{\omega C} = \frac{1}{2500(40\mu f)} = 10\Omega$$

$$\bar{Z} = R - jXc$$
$$= 10 - j10 = 14.14\measuredangle -45°$$

$$\bar{V} = \frac{V_p}{\sqrt{2}} = \frac{500}{\sqrt{2}} = 353.6v\measuredangle -20°$$

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{353.6v\measuredangle -20°}{14.14\measuredangle -45°} = 25A\measuredangle 25°$$

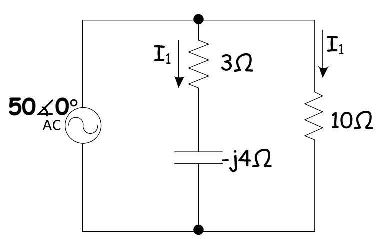
$$i(t) = 25A\cos(2500t + 25°)$$

(Current is LEADING Voltage as would be expected in a Capacitive circuit)

Example 2:

(a) For the parallel circuit shown below, find the **branch currents** and the **total current**.

(b) Find Z_{eq} from V/I and compare with the result of $(Z_1 Z_2)/(Z_1 + Z_2)$



$$I_1 = \frac{V}{Z_1} = \frac{50\measuredangle 0°}{3 - j4} = \frac{50\measuredangle 0°}{5\measuredangle -53.13°} = \boxed{10A\measuredangle 53.13°}$$

$$I_2 = \frac{V}{Z_2} = \frac{50\measuredangle 0°}{10} = \boxed{5A\measuredangle 0°}$$

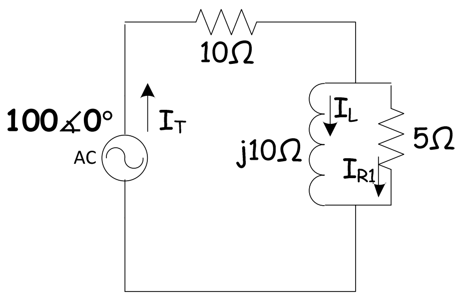
By KCL

$$0 = -I_T + I_1 + I_2$$
$$I_T = I_1 + I_2 = 10A\measuredangle 53.13° + 5A\measuredangle 0°$$
$$= 6 + j8 + 5 = \boxed{11 + j8} = \boxed{13.6A\measuredangle 36°}$$

$$Z_T = \frac{V_T}{I_T} = \frac{50V\measuredangle 0°}{13.6A\measuredangle 36°} = \boxed{3.676\Omega\measuredangle -36°}$$

$$Z_T = Z_1 \,||\, Z_2$$
$$= \frac{(5\measuredangle -53.13°)(10\measuredangle 0°)}{3 - j4 + 10} = \frac{50\measuredangle -53.13°}{13.6\measuredangle -17.1°} = \boxed{3.676\Omega\measuredangle -36°}$$

Example 3: For the **series-parallel circuit** shown below, find the **current** in each element.



$$Z_T = 10\Omega + \frac{5\Omega\left(j10\Omega\right)}{5\Omega + j10\Omega}$$

$$= 10\Omega + \frac{50\Omega\measuredangle 90^\circ}{11.18\measuredangle 63.43^\circ}$$

$$= 10\Omega + \left(\frac{50\Omega\measuredangle 90^\circ}{11.18\measuredangle 63.43^\circ}\right)$$

$$= 10 + \left(4 + j2\right) = 14 + j2 = 14.14\measuredangle 8.13^\circ$$

$$I_T = \frac{100\measuredangle 0^\circ}{14.14\measuredangle 8.13^\circ} = \boxed{7.07\measuredangle -8.13^\circ}$$

$$I_L = \frac{I_T R_1}{R_1 + Z_L} = \frac{7.07\measuredangle -8.13^\circ\left(5\right)}{5 + j10}$$

$$= \frac{35.36\measuredangle -8.13^\circ\left(5\right)}{11.18\measuredangle 63.43^\circ} = \boxed{3.162\measuredangle -71.56^\circ}$$

$$IR_1 = I_T - I_L$$

$$= 7.07\measuredangle -8.13^\circ - \left(3.162\measuredangle -71.56^\circ\right)$$

$$= \left(7 - j1\right) - \left(1 - j3\right) = 6 + j2 = \boxed{6.325\measuredangle 18.43^\circ}$$

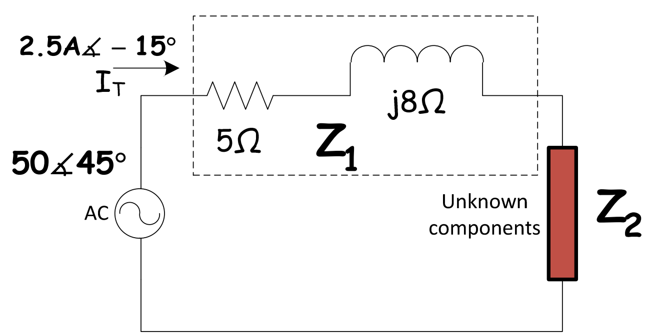
Problem 1: The series circuit of **R = 8 ohms** and **L = 0.02 Henrys** has an applied voltage of:

$v(t) = 283\sin(300t + 90^\circ)$. **Find the current i(t).**

$$
\begin{aligned}
v(t) &= 283\sin(300t + 90^\circ) \\
&= 283\cos(300t + 90^\circ + 90^\circ) \\
&= 283\cos(300t + 180^\circ) \\
\bar{V} &= \frac{283}{\sqrt{2}}\measuredangle 180^\circ = 200.1\measuredangle 180^\circ \ @\ \omega = 300\ \text{rad}/\text{s} \\
X_L &= \omega L = 300(0.02) = 6\Omega \\
\bar{I} &= \frac{\bar{V}}{Z_T} = \frac{200.1V\measuredangle 180^\circ}{8\Omega + j6\Omega} = \frac{200.1V\measuredangle 180^\circ}{10\measuredangle 36.87^\circ} = \boxed{20.01\measuredangle 143.1^\circ} \\
i(t) &= 20.01\sqrt{2}\cos(300t + 143.1^\circ) \\
&= 28.3\sin(300t + 143.1^\circ - 90^\circ) = \boxed{28.3\sin(300t + 53.1^\circ)}
\end{aligned}
$$

Of course, knowing where we were going to end up, you could have left out several steps including the conversion to COS (since we were going to end up back at SIN anyway).

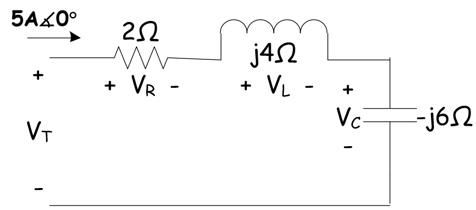
Problem 2: Calculate the impedance, Z_2, in the series circuit shown below:



From the given current,

$$
\begin{aligned}
Z_T &= \frac{50\measuredangle 45^\circ}{2.5\measuredangle -15^\circ} = 20\measuredangle 60^\circ = 10 + j17.32 \\
&= Z_1 + Z_2 \\
Z_2 &= Z_T - Z_1 \\
&= 10\Omega + j17.32\Omega - (5\Omega + j8\Omega) \\
&= \boxed{5\Omega + j9.32\Omega} = \boxed{10.58\Omega\measuredangle 61.79^\circ}
\end{aligned}
$$

Problem 3: In the series circuit shown below, the **effective (rms) value of the current** is **5 amps**. What are the readings on the voltmeter placed 1st across the entire circuit and then across each element?

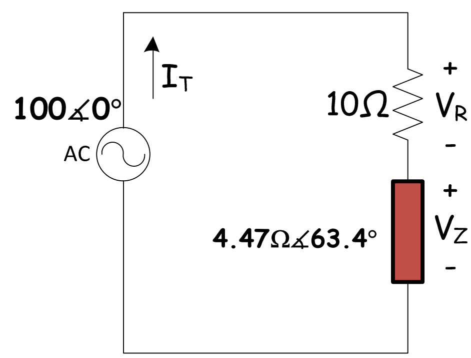


Remember that voltmeters do not measure phase angle

$$\begin{aligned}\bar{V}_T &= I_T Z_T \\ &= 5A\measuredangle 0^\circ (2\Omega + j4\Omega - j6\Omega) \\ &= 5A\measuredangle 0^\circ (2\Omega - j2\Omega) \\ &= 10 - j10 = 14.14\measuredangle 45^\circ \\ &= \boxed{14.14V}\end{aligned}$$

$$\begin{aligned}\bar{V}_R &= I_T R \\ &= 5A\measuredangle 0^\circ (2\Omega) = \boxed{10V} \\ \bar{V}_L &= I_T Z_L \\ &= 5A\measuredangle 0^\circ (4\Omega\measuredangle 90^\circ) = \boxed{20V} \\ \bar{V}_C &= I_T Z_C \\ &= 5A\measuredangle 0^\circ (6\Omega\measuredangle -90^\circ) = \boxed{30V}\end{aligned}$$

Problem 4: The two impedances $\mathbf{Z_1}$ and $\mathbf{Z_2}$ shown are in series with a voltage of $V = 100\measuredangle 0^\circ$. Find the voltage across each impedance.



$$\begin{aligned}I_T &= \frac{V}{Z_T} = \frac{100v\measuredangle 0^\circ}{10 + 4.47\measuredangle 63.4^\circ} \\ &= \frac{100v\measuredangle 0^\circ}{10 + (2 + j4)} = \frac{100v\measuredangle 0^\circ}{12 + j4} \\ &= \frac{100v\measuredangle 0^\circ}{12.65\measuredangle 18.42^\circ} = \boxed{7.9A\measuredangle -18.42^\circ} \\ V_R &= I_T R = (7.9A\measuredangle -18.42^\circ)\,10\Omega \\ &= \boxed{79V\measuredangle -18.42^\circ} = \boxed{74.9V - j25V} \\ V_Z &= I_T Z_2 \\ &= (7.9A\measuredangle -18.42^\circ)\,4.47A\measuredangle 63.4^\circ \\ &= \boxed{35.31V\measuredangle 45^\circ} = \boxed{25V + j25V}\end{aligned}$$

Check :

$$\begin{aligned}VR + VZ &= (74.9V - j25V) + (25V + j25V) \\ &= 99.9V \approx 100V\end{aligned}$$

The Power Triangle

Real Power is defined as:

$$\boxed{\begin{aligned} P &= \frac{1}{2}E_pI_p\cos\theta \quad \text{WATTS}(W) \\ &= E_{rms}I_{rms}\cos\theta \end{aligned}}$$

Where θ is the **Power Factor angle** (it also turns out to be the impedance angle). The **Power Factor angle** is measured **from the voltage phasor to the current phasor**.

The **Power Factor (pf)** itself is defined as: $pf = \cos\theta$ The terms **leading** and **lagging** are usually attached to the Power Factor. These terms are **defined by the current thru the load**.

- If the **current waveform thru the load is leading the voltage waveform** across the load then the **power factor is a LEADING pf**.
- If the **current waveform thru the load is lagging the voltage waveform** across the load then the **power factor is a LAGGING pf**.

Reactive power is defined as:

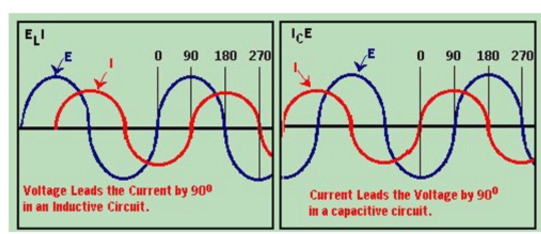
$$\boxed{\begin{aligned} Q &= \frac{1}{2}E_pI_p\sin\theta \quad \text{Volt-Amperes Reactive } (\text{VARS}) \\ &= E_{rms}I_{rms}\sin\theta \end{aligned}}$$

Apparent power is defined as:

$$\boxed{\begin{aligned} S &= \frac{1}{2}E_pI_p \quad \text{Volt-Amperes } (\text{VA}) \\ &= E_{rms}I_{rms} \\ &= P \pm jQ \end{aligned}}$$

For **totally resistive** circuits, $\theta = 0$, so **Real power** is:

$$P = E_{rms}I_{rms} = \frac{E_{rms}^2}{R_T} = I_{rms}^2R_T$$



ELI the **ICE** person
ELI➔Voltage Leads **Current** in **Inductive** circuits (or **Current** Lags **Voltage** since the current phase angle is negative),
ICE➔Current Leads **Voltage** in **Capacitive** circuits (since the current phase angle is positive)

POWER IN AC CIRCUITS Example

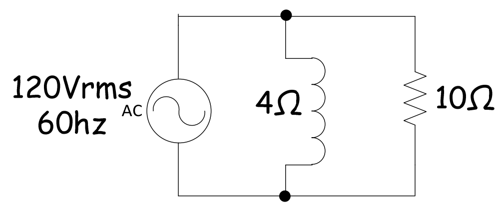
For the circuit shown, determine the:

a) Apparent power

b) Real power

c) Reactive power

d) Power Triangle



SOLUTION

$$Z_T = \frac{10(4\angle 90^\circ)}{10 + j4} = \frac{40\angle 90^\circ}{10.77\angle 21.8^\circ} = 3.714\angle 68.2^\circ\Omega$$

$$P.F = \cos\theta = \cos(68.2^\circ) = .3713$$

$$I_T = \frac{E_T}{Z_T} = \frac{120\text{Vrms}}{3.714\Omega\angle 68.2^\circ} = 32.31\angle -68.2^\circ A$$

$$\text{Apparent Power} = S = E_{rms}I_{rms} = 120V(32.31A) = \boxed{3.877kVA}$$

$$\text{Real Power} = P = E_{rms}I_{rms}\cos\theta = S\cos 68.2^\circ = 3.877kVA(.3713) = \boxed{1.44kW}$$

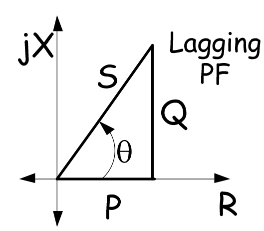
or

$$\text{Real Power} = P = \frac{V_R^2}{R} = \frac{(120V_{rms})^2}{10\Omega} = \boxed{1.44kW}$$

$$\text{Reactive Power} = Q = E_{rms}I_{rms}\sin\theta = S\sin 68.2^\circ = 3.877kVA(.9285) = \boxed{3.6kVAR}$$

or

$$\text{Reactive Power} = Q = \frac{V_L^2}{X_L} = \frac{(120V_{rms})^2}{4\Omega} = \boxed{3.6kVAR}$$



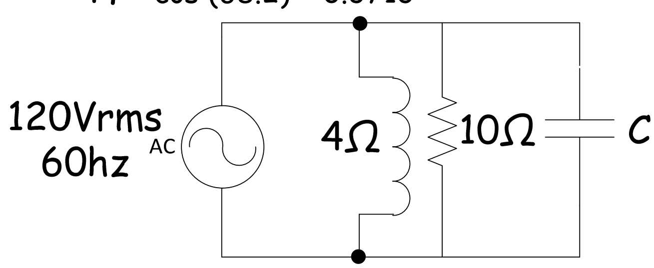
POWER FACTOR CORRECTION

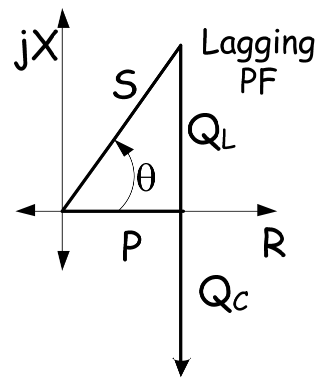
- **Apparent Power** is what is paid for but **Real Power** is what is dissipated. Another way of looking at it is that the smaller the power factor is (the farther away from 1), the less efficient the system is. It is normal practice for the power company to add a penalty to a company's electric bill based on **how far away from '1'** the power factor is.
- It is common practice by industry to perform **Power Factor Correction** to offset the reactive power, thus bringing Apparent Power in line with Real Power and the amount of **actual current down.** This is done by placing a capacitive load in parallel with the factory's Inductive load to counteract the inductive reactive power.

Example: Continue the last example by calculating the size of capacitor which would be required to bring the power factor to 1.

Known (from the last example)

- **P** = 1.44**kW**
- **Q** = 3.6**KVAR**
- **S** = 3.877**KVA**
- **Pf** = cos (68.2) = 0.3713





$$Q_C = -Q_L = -3.6\text{kVAR}$$

$$Q_C = \frac{E^2}{X_C}$$

$$X_C = \frac{E^2}{Q_C} = \frac{(120\text{Vrms})^2}{3.6\text{kVAR}} = 4\Omega$$

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(60\text{hz})4\Omega} = \boxed{663.1\mu\text{F}}$$

$$I_{uncorrected} = \frac{S}{E} = \frac{3.877\text{kVA}}{120\text{V}} = 32.31\text{A}$$

$$\Rightarrow Z_T = \frac{1}{\frac{1}{10\Omega} + \frac{1}{4\Omega\angle 90^\circ} + \frac{1}{4\Omega\angle -90^\circ}} = 10\angle 0^\circ$$

$$pf = \cos(0^\circ) = 1$$

$$I_{corrected} = \frac{S}{E} = \frac{P}{E} = \frac{1.44\text{kVA}}{120\text{V}} = 12\text{A}$$

Substantially lower current!!!!

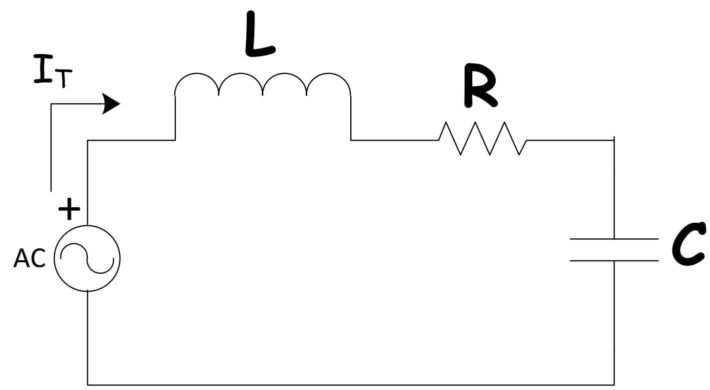
RESONANCE

Up to this point, any sources we had in a circuit were required to have a **CONSTANT FREQUENCY**. In fact, it is a very helpful when solving for voltages and currents to be able to assume that only the phase angle will change. The Frequency has always stayed constant. As a matter of fact, once you converted into the **PHASOR DOMAIN**, all that the frequency was needed for was for calculating inductive and capacitive reactance.

The study of **resonance** is the study of AC **circuit behavior when the frequency of the AC sources is allowed to change**.

SERIES RESONACE

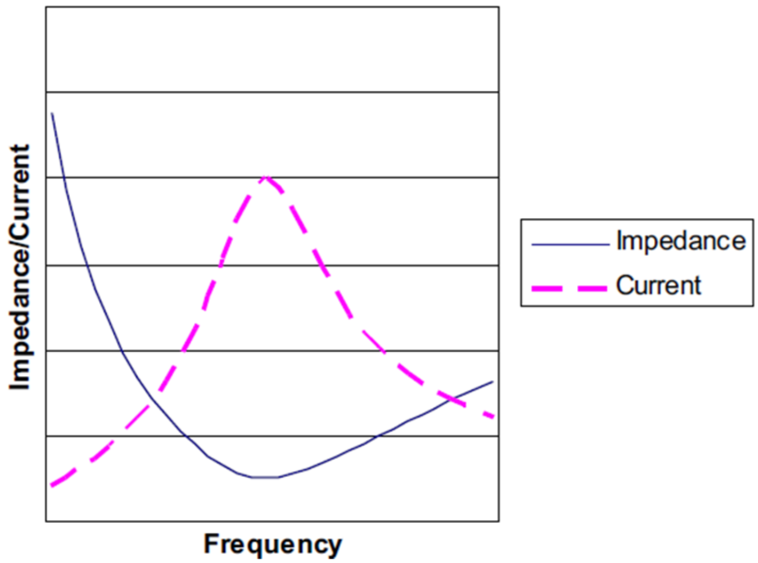
The circuit shown here consists of a **source**, an **R, L,** and **C**. We assume that the source peak value will remain constant, but the frequency will be allowed to vary. Now we need to develop and understanding of how the voltages, currents, and impedances will vary with the frequency and after being exposed to each other.



The impedance of the circuit (in the time domain) is simply: $\mathbf{Z} = \mathbf{R} + \mathbf{j}\mathbf{X_L} - \mathbf{j}\mathbf{X_C}$. If we were to backtrack and substitute the frequency equivalents we would get:

$\mathbf{Z} = \mathbf{R} + \mathbf{j}\left(\omega\mathbf{L} - \frac{\mathbf{1}}{\omega\mathbf{C}}\right)$, and the current equation: $\mathbf{I} = \dfrac{\mathbf{V}}{\mathbf{R} + \mathbf{j}\left(\omega\mathbf{L} - \frac{\mathbf{1}}{\omega\mathbf{C}}\right)}$.

A plot of the MAGNITUDE of impedance and current as vs. frequency is shown to be:



There comes a point on the graph where the capacitive and inductive reactance's CANCEL each other out: $\omega \mathbf{L} = \dfrac{1}{\omega \boldsymbol{C}}$.

When this cancellation occurs in a **SERIES RESONANT circuit**, the:

- **Impedance** goes through a **MINIMUM** and
- The **current** goes through a **MAXIMUM**.

Of course, this only occurs at a specific frequency known as the "**SERIES RESONANT FREQUENCY**".

$$\omega_r = \frac{1}{\sqrt{LC}} \quad \text{or} \quad f_r = \frac{1}{2\pi\sqrt{LC}}$$

EXAMPLE PROBLEM:
Determine the **SERIES RESONANT FREQUENCY (in hertz)** if **C = 0.02μF, L = 0.1H, R = 100 ohms** and the source voltage is **60v∡0°**. Also determine the **maximum current at resonance.**

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(0.1H)(0.02\mu f)}} = 3.559khz \simeq \boxed{3.56khz}$$

$$I_r(max) = \frac{V}{R} = \frac{60v}{100\Omega} = 600mA\left[\text{since the reactances cancelled}\right]$$

Quality Factor (Q)

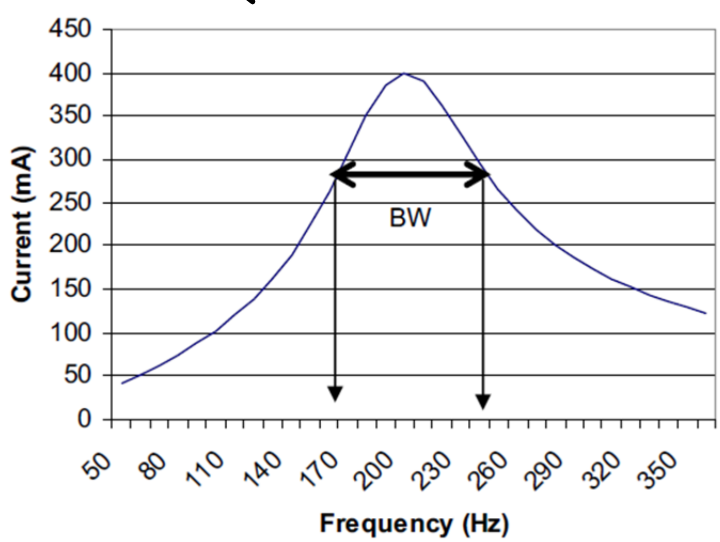
A very important characteristic of resonance is the width of the curve around the resonance frequency. This width is called the **Quality Factor**, or more commonly referred to as **Q**.

> *The greater the Q, the sharper and narrower the resonant curve (the more selective it becomes)*

The **Q** of a **Series Resonant Circuit** is given by: $Q_s = \dfrac{\omega_r L}{R}$.

BANDWIDTH (BW)

Another important measure of the resonant curve is **BANDWIDTH (BW).** This bandwidth is measured in hertz and is the **difference in frequencies between the half-power points** on the curve. The BW is given by: $BW = \dfrac{f_r}{Q}$



The **half-power points** occur when the **RMS** value is at: $\dfrac{I_{rms}}{\sqrt{2}}$

For this graph, the actual **BW**: **250 Hz – 180 Hz = 70 Hz**.

SERIES RESONANCE EXAMPLE PROBLEM:

Consider a **series** circuit with the following values: **L =54mH and C = 470pF**.

A) What is the **resonant frequency**?

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{\left(54 \bullet 10^{-3}H\right)\left(4700 \bullet 10^{-12}F\right)}} = 9.99kHz \simeq \boxed{10kHz}$$

B) What value of **R** is required to provide a **BW of 100 Hz**?

$$\boxed{\begin{array}{l} Q_s = \dfrac{\omega_r L}{R} \\ BW = \dfrac{f_r}{Q} \end{array} \quad \text{and} \quad f_r = 10khz}$$

$$R = \frac{\omega_r L}{Q} = \frac{2\pi f_r L}{Q}$$

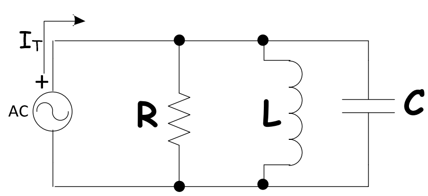
$$Q = \frac{f_r}{BW}$$

$$\text{so,} \quad R = \frac{2\pi f_r L}{\frac{f_r}{BW}} = \frac{2\pi \cancel{f_r} L(BW)}{\cancel{f_r}}$$

$$R = 2\pi L(BW) = 2\pi(54mH)(100Hz) = \boxed{33.93\Omega}$$

PARALLEL RESONANCE

A second type of **RESONACE** is **PARALLEL RESONANCE**. Rather than go thru the math with this type we will just focus on the equations and <u>**the effect on the circuit**</u> at **RESONANCE**. Some of the equations are the same as for the SERIES RESONANCE circuits.



$$\omega_r = \frac{1}{\sqrt{LC}}$$

$$Q_p = \frac{R}{\omega L}$$

Note the difference in this equation for Q_p from the equation for Q_s in SERIES RESONANCE.

$$BW = \frac{f_r}{Q}$$

PARALELL RESONANCE EXAMPLE:

For a parallel circuit with **R = 100kΩ, C = .001μF, L = 15mH**, $\overline{\mathbf{V}} = 20\measuredangle 0^\circ$:

a) What is the resonant frequency, f_r?

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(15mH)(.001\mu F)}} = \boxed{41.09kHz}$$

b) What if the **Q** and the **BW**?

$$Q = \frac{R}{\omega_r L} = \frac{R}{2\pi f_r L} = \frac{100k\Omega}{2\pi(41.09kHz)(15mH)} = \boxed{25.82}$$

$$BW = \frac{f_r}{Q} = \frac{41.09kHz}{25.82} = \boxed{1.59kHz}$$

c) What is the **current of from the source** at **resonance**? $I = \dfrac{V}{R} = \dfrac{20v}{100k\Omega} = \boxed{200\mu A}$

RESONANCE REVIEW

- **Capacitive reactance decreases** as the applied **frequency increases**
- **Inductive reactance increases** as the applied **frequency increases**
- At some specific frequency $X_C = X_L$ and they will cancel out.

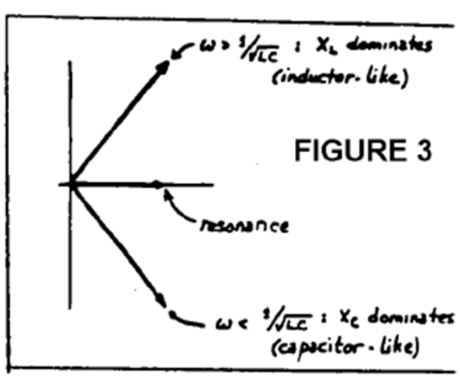
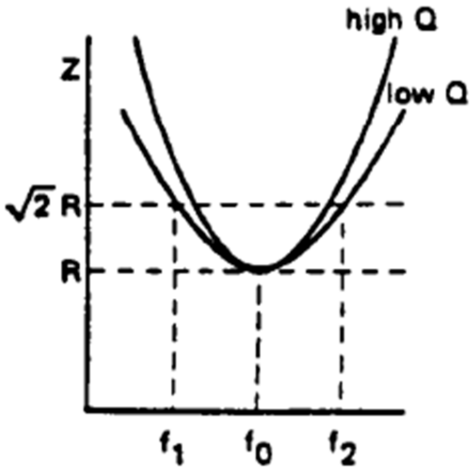
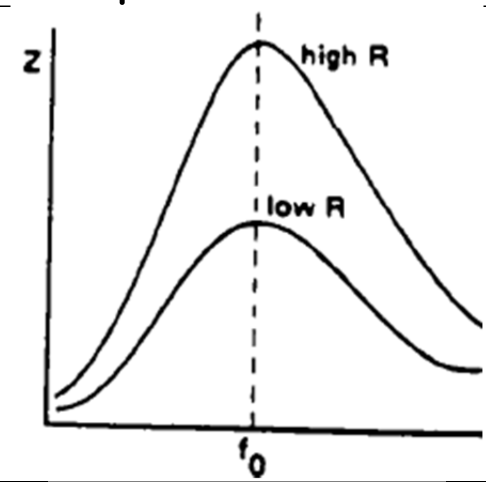
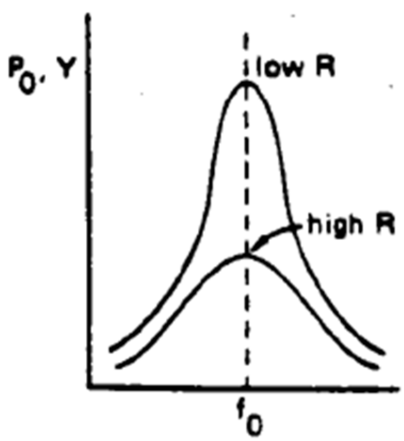


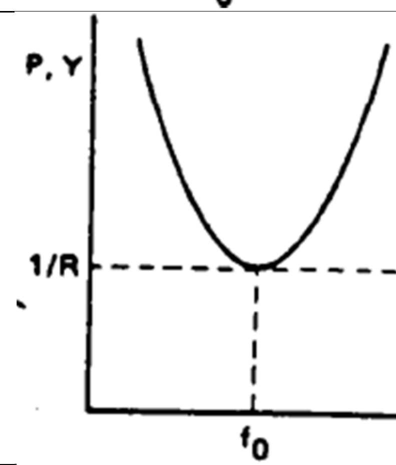
FIGURE 3

SERIES RESONANCE	PARALLEL RESONANCE
Impedance is MINIMUM **Impedance = R** **Current and Voltage are in phase** **Current is at a MAXIMUM** **Power dissipation is MAXIMUM**	**Impedance is MAXIMUM** **Impedance = R** **Current and Voltage are in phase** **Current is at a MINIMUM** **Power dissipation is MINIMUM**









Series Resonance	Common Equations	Parallel Resonance
$Q = \frac{1}{\omega_o RC} = \frac{\omega_o L}{R}$	$\omega_o = \frac{1}{\sqrt{LC}}$ $Z = R$ $BW = \omega_o / Q$	$Q = \omega_o RC = \frac{R}{\omega_o L}$